Problem A1 (Electrostatics):

An infinite, linear dielectric slab of thickness d and permittivity ε occupies the region $-d/2 < z < d/2$. The slab is placed in an otherwise uniform electric field

$$\mathbf{E}_0 = E_0\hat{\mathbf{z}}.$$

Assume vacuum outside the slab.

1. Determine the electric field $\mathbf{E}(z)$everywhere.
2. Find the bound surface charge densities at $z = \pm d/2$.
3. Compute the polarization $\mathbf{P}$ inside the slab and verify consistency with boundary conditions.

Solution

1. Because the system is planar and infinite, the field depends only on $z$and is along $\hat{\mathbf{z}}$. There are no free charges anywhere, so the normal component of $\mathbf{D}$ is continuous:

$$D_z^{(\mathrm{in})} = D_z^{(\mathrm{out})}.$$

Outside (vacuum):

$$\mathbf{D}_{\mathrm{out}} = \varepsilon_0\mathbf{E}_{\mathrm{out}}.$$

Inside:

$$\mathbf{D}_{\mathrm{in}} = \varepsilon\mathbf{E}_{\mathrm{in}}.$$

Thus,

$$\varepsilon E_{\mathrm{in}} = \varepsilon_0 E_{\mathrm{out}}.$$

Type equation here.Far away, the field must approach the applied field:

$$\mathbf{E}_{\mathrm{out}} = E_0\hat{\mathbf{z}}.$$

Hence,

$$\boxed{\mathbf{E}_{\mathrm{in}} = \frac{\varepsilon_0}{\varepsilon}E_0\hat{\mathbf{z}}}$$

$$\boxed{\mathbf{E}_{\mathrm{out}} = E_0\hat{\mathbf{z}}}$$

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2. Polarization inside the slab is

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}_{\mathrm{in}}.$$

Thus,

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\frac{\varepsilon_0}{\varepsilon}E_0\hat{\mathbf{z}}.$$

Bound surface charge:

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}.$$

At $z = +d/2$, $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$:

$$\boxed{\sigma_b(+d/2) = P = (\varepsilon - \varepsilon_0)\frac{\varepsilon_0}{\varepsilon}E_0}$$

At $z = -d/2$, $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$:

$$\boxed{\sigma_b(-d/2) = -P = -(\varepsilon - \varepsilon_0)\frac{\varepsilon_0}{\varepsilon}E_0}$$

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3. Inside:

$$\boxed{\mathbf{P} = \varepsilon_0(\varepsilon_r - 1)\mathbf{E}_{\text{in}}}$$

Check boundary condition for **D**:

$$D_z^{\text{out}} - D_z^{\text{in}} = \sigma_{\text{free}} = 0$$

which is satisfied since

$$\varepsilon_0 E_0 = \varepsilon E_{\text{in}}.$$

Check discontinuity of **E**:

$$E_z^{\text{out}} - E_z^{\text{in}} = \frac{\sigma_b}{\varepsilon_0}$$

which is also satisfied using the above expressions.

Problem B1 (Electrostatics):

A grounded conducting sphere of radius $R$is centered at the origin. A point charge $q$is placed on the z-axis at position $z = a$, where $a > R$.

1. Use the method of images to find the electrostatic potential outside the sphere.
2. Determine the magnitude and position of the image charge.
3. Find the total induced charge on the conducting sphere.
4. Find the force acting on the point charge q.

Solution

Because the sphere is grounded, its surface is held at zero potential:

$$V(R,\theta) = 0.$$

The method of images replaces the grounded sphere by an image charge q'placed inside the sphere on the z-axis at position $z = b$. The potential outside the sphere is written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{r} - a\hat{\mathbf{z}}|} + \frac{q'}{|\mathbf{r} - b\hat{\mathbf{z}}|}\right], \qquad r \geq R.$$

We choose $q'$and $b$so that $V = 0$on $r = R$.

For a point on the sphere,

$$|\mathbf{r} - a\hat{\mathbf{z}}| = \sqrt{R^2 + a^2 - 2aR\cos\,\theta},$$

and

$$|\mathbf{r} - b\hat{\mathbf{z}}| = \sqrt{R^2 + b^2 - 2bR\cos\,\theta}.$$

The correct image charge is

$$\boxed{q' = -q\frac{R}{a}}$$

located at

$$\boxed{b = \frac{R^2}{a}.}$$

Thus the potential outside the sphere is

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{r} - a\hat{\mathbf{z}}|} - \frac{qR/a}{|\mathbf{r} - (R^2/a)\hat{\mathbf{z}}|}\right], r \geq R.}$$

Inside the conductor,

$$\boxed{V = 0.}$$

The total induced charge on the grounded conducting sphere equals the total image charge:

$$\boxed{Q_{\text{ind}} = q' = -q\frac{R}{a}.}$$

This is negative if $q > 0$, as expected.

The force on the real charge q is the Coulomb force due to the image charge q'. The separation between $q$and q' is

$$a - b = a - \frac{R^2}{a} = \frac{a^2 - R^2}{a}.$$

Therefore,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|\, qq' \,|}{(a-b)^2}.$$

Substituting $q' = -qR/a$,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2R/a}{\left(\frac{a^2 - R^2}{a}\right)^2}.$$

Thus,

$$\boxed{F = \frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2 - R^2)^2}.}$$

The force is attractive, directed toward the sphere:

$$\boxed{\mathbf{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2 - R^2)^2}\hat{\mathbf{z}}.}$$

Problem B3 (Magnetostatics):

A thin spherical shell of radius R carries a uniform surface charge density σ. The shell rotates with constant angular velocity

$$\boldsymbol{\omega} = \omega \hat{\mathbf{z}}.$$

Assume the rotation is nonrelativistic, so electrostatic charge redistribution can be neglected.

1. Find the surface current density **K**.
2. Using the Biot–Savart law, find the magnetic field at the center of the sphere.

Solution

1. A point on the surface moves with velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Since $\mathbf{r} = R\hat{\mathbf{r}}$,

$$\mathbf{v} = \omega R \sin\,\theta\,\hat{\boldsymbol{\phi}}.$$

The surface current density is charge per unit area times velocity:

$$\mathbf{K} = \sigma \mathbf{v}.$$

Therefore,

$$\boxed{\mathbf{K} = \sigma\omega R \sin\,\theta\,\hat{\boldsymbol{\phi}}.}$$

2. Now calculate the magnetic field at the center. Consider a narrow circular strip of the spherical shell between $\theta$and $\theta + d\theta$. Its radius is

$$a = R\sin\,\theta,$$

and its distance from the center along the z-axis is

$$z = R\cos\,\theta.$$

The width of the strip is

$$R\,d\theta.$$

Thus the current carried by this strip is

$$dI = K\,R\,d\theta = \sigma\omega R^2 \sin\,\theta\,d\theta.$$

The magnetic field on the axis of a circular current loop is

$$dB_z = \frac{\mu_0 dI\,a^2}{2(a^2 + z^2)^{3/2}}.$$

Here,

$$a^2 + z^2 = R^2.$$

Therefore,

$$dB_z = \frac{\mu_0 dI\, R^2 \sin^2\theta}{2R^3} = \frac{\mu_0 dI}{2R}\sin^2\theta.$$

Substituting dI,

$$dB_z = \frac{\mu_0 \sigma \omega R}{2}\sin^3\theta\, d\theta.$$

Integrating over the full sphere,

$$B_z(0) = \frac{\mu_0 \sigma \omega R}{2}\int_0^{\pi}\sin^3\theta\, d\theta.$$

Since

$$\int_0^{\pi}\sin^3\theta\, d\theta = \frac{4}{3},$$

we obtain

$$\boxed{\mathbf{B}(0) = \frac{2}{3}\mu_0 \sigma \omega R\, \hat{\mathbf{z}}.}$$

The field points along $+\hat{\mathbf{z}}$ for $\sigma > 0$ and $\omega > 0$.

Problem B2 (Circuits):

A resistor $R$and capacitor $C$are connected in series with a battery of emf V_0. At time $t = 0$, the switch is closed and the capacitor is initially uncharged.

1. Find the current $I(t)$in the circuit.
2. Find the charge $Q(t)$on the capacitor.
3. Find the voltage across the capacitor $V_C(t)$.
4. Determine the characteristic time scale of the charging process.
5. Find the energy stored in the capacitor after a long time.

Solution

Using Kirchhoff's loop rule,

$$V_0 - IR - \frac{Q}{C} = 0.$$

Since

$$I = \frac{dQ}{dt},$$

the equation becomes

$$R\frac{dQ}{dt} + \frac{Q}{C} = V_0.$$

Rearranging,

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{V_0}{R}.$$

Solve with the initial condition

$$Q(0) = 0.$$

The solution is

$$\boxed{Q(t) = CV_0\left(1-e^{-t/RC}\right).}$$

The current is

$$I(t) = \frac{dQ}{dt},$$

therefore

$$\boxed{I(t) = \frac{V_0}{R}e^{-t/RC}.}$$

The voltage across the capacitor is

$$V_C(t) = \frac{Q(t)}{C},$$

so

$$\boxed{V_C(t) = V_0\left(1-e^{-t/RC}\right).}$$

The characteristic charging time is

$$\boxed{\tau = RC.}$$

As $t \to \infty$,

$$Q \to CV_0, V_C \to V_0.$$

The energy stored in the capacitor is

$$U = \frac{1}{2} C V_0^2.$$

Thus,

$$\boxed{U = \frac{1}{2} C V_0^2.}$$

Problem A2

A conducting rod of length $L = 0.50$ m slides on two parallel rails with a constant speed of $v = 8.0$ m/s. The rails are connected through a resistor of resistance $R = 4.0\ \Omega$. A uniform magnetic field of magnitude $B = 0.30$ T points perpendicular to the plane of the rails. What external power is required to keep the rod moving at constant speed?

Solution:

For a rod moving perpendicular to a magnetic field,

$$\mathcal{E} = BLv$$

So: $\mathcal{E} = (0.30)(0.50)(8.0) = 1.2$ V

The magnetic force on a current-carrying conductor is:

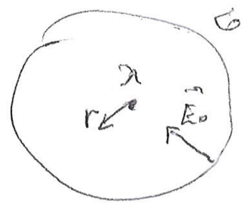
$$F = BIL$$

So: $F = (0.3)(0.3)(0.5) = 0.045$ N

Now to maintain constant speed, the external force must balance the magnetic force.

Power supplied: $P = Fv = (0.045)(8.0) = 0.36$ W

EM A3.



l.h.s. 5
r.h.s. 5

use the Gauss' law inside the cylinder

$-E \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0}$ where l is the length of a section of the wire

$(E > 0)$

$-2\pi r E = \frac{\lambda}{\varepsilon_0}$

3 for $r = R$ $\quad \underline{\lambda = -2\pi\varepsilon_0 R E_0 < 0}$

Boundary condition on the cylinder surface

5 $E(R+0) - E(R-0) = \frac{\sigma}{\varepsilon_0}$

5 since $E(R+0) = 0$, $\quad E(R-0) = -E_0$

$\underline{\sigma = -\varepsilon_0 E(R-0) = +\varepsilon_0 E_0} > 0$

2

The same can be obtained from the Gauss' law outside the cylinder

EM A4

From $\vec{\nabla}\times\vec{B} = \mu_0 j\,\hat{x}$

4 $$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 j_0 (y^2+z^2) \quad (1)$$

3 $$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

3 $$\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = 0$$

also 4 $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

From Eq. (1) we guess the solution

5 $$B = \frac{\mu_0 j_0}{3}\left(y^3\hat{z} - z^3\hat{y}\right) \quad (2)$$

now check the rest of the equations

3 $$\frac{\partial B_y}{\partial x} = 0, \quad \frac{\partial B_x}{\partial y} = 0, \quad \frac{\partial B_z}{\partial x} = 0, \quad \frac{\partial B_x}{\partial z} = 0$$

$$\frac{\partial B_x}{\partial x} = 0, \quad \frac{\partial B_y}{\partial y} = 0, \quad \frac{\partial B_z}{\partial z} = 0$$

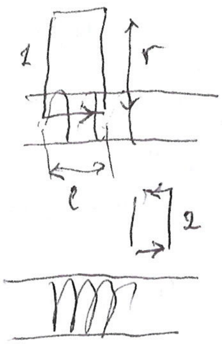
Therefore (2) is the solution indeed

calculation $y = 2m$, $z = 1m$

3 $$\vec{B} = \frac{4\pi\times10^{-7}\cdot 200}{3}\left(8\hat{z} - 1\cdot\hat{y}\right) = 8.38\times10^{-5}\left(8\hat{z}-\hat{y}\right)\,T$$

$B_y = -8.38\times10^{-5}\,T$, $B_z = 6.70\times10^{-4}\,T$, $B_x = 0$

EM B4



1. From the Ampere law

For the loop 1

$$B \cdot l = \mu_0 I_{enc} \quad \text{Since } B(r) \to 0 \text{ if } r \to \infty$$

$$I_{enc} = n l I$$

4

$$B_{in} = \mu_0 n I$$

For the loop 2

2 $$B \cdot l = I_{enc} = 0 \to B_{out} = 0$$

2. Use $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$ where Φ is the magnetic flux

4

$\vec{E}$ has only $\hat{\phi}$ component

2 $$\oint \vec{E} \cdot d\vec{l} = 2\pi r E$$

$$\Phi = NB \cdot \pi r^2 \quad \text{inside}$$

3 $$NB \cdot \pi R^2 \quad \text{outside}$$

Therefore $E = \frac{N\dot{B}r}{2}$ inside, where $\dot{B} = \frac{dB}{dt} = \mu_0 n \dot{I}$

2 $$= \frac{N\dot{B}R^2}{2r} \quad \text{outside}$$

3. In calculating B we used only conduction current, ~~But~~ with the account of the displacement current

3 $$\vec{J}_{tot} = \varepsilon_0 \frac{\partial E}{\partial t} + \vec{J}$$

The above treatment is valid if $\varepsilon_0 \frac{\partial E}{\partial t} \ll j$

3 $$E_{max} = \frac{N\dot{B}R}{2} = \mu_0 \frac{nN\dot{I}R}{2} \to \varepsilon_0 \mu_0 \frac{nN\ddot{I}R}{2} \ll \frac{I}{\pi R^2}$$

using $\varepsilon_0\mu_0 = 1/c^2$, we obtain

2 $$\left|\frac{\ddot{I}}{I}\right| \ll \frac{2c^2}{nN\pi R^3}$$

using $\left|\frac{\ddot{I}}{I}\right| = \omega^2 = 10^4 s^{-1}$, $N = 100$, $n = 10^3 m^{-1}$, $R = 0.01 m$

we obtain $$10^4 \ll \frac{2 \cdot 9 \cdot 10^{16}}{10^3 \cdot 10^2 \cdot \pi \cdot 10^{-6}}$$

which is satisfied very well

QM (A1) $e^- + h\nu \to e^-$

conservation of energy

5 (1) $mc^2 + h\nu = m\gamma c^2 \qquad \gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$

conservation of momentum

5 (2) $\frac{h\nu}{c} = mv\gamma$

5 from (1) $\frac{h\nu}{c} = mc(\gamma-1)$

and from (2)

$$mc(\gamma-1) = mv\gamma$$

5 $$\gamma = \frac{1}{1-\frac{v}{c}} = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

5 which is possible only if $v=0, \nu=0$

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in case they do nonrelativistic version (partial credit)

4 $h\nu = \frac{mv^2}{2}$

4 $\frac{h\nu}{c} = mv$

3 $\frac{mv^2}{2c} = mv$

From here either $v = \nu = 0$

2 or $v = 2c$ which is not possible

QM (A2)

$$\psi = N e^{ikx-\alpha x^2-i\omega t}$$

2 prob. density $\rho = |\psi|^2 = N^2 e^{-2\alpha x^2}$

5 current density $j = \frac{\hbar}{m} N^2 \operatorname{Im}\left[e^{-ikx-\alpha x^2+i\omega t} \cdot \frac{d}{dx}\left(e^{ikx-\alpha x^2-i\omega t}\right)\right]$

calculate 8 $= \frac{\hbar}{m} N^2 \operatorname{Im}\left[e^{-ikx-\alpha x^2+i\omega t}(ik-2\alpha x)\, e^{ikx-\alpha x^2-i\omega t}\right]$

$$= \frac{\hbar}{m} N^2 e^{-2\alpha x^2} k$$

continuity equation

5 $$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

in our case $\frac{d\rho}{dt} = 0$, $\frac{\partial j}{\partial x} \neq 0$

5 not satisfied, therefore ψ is not a solution to the Sch. eq.

QM (A3)

quantum effects are important

3+3 if $\lambda > r_s$ where $r_s \sim \frac{1}{n^{1/3}}$

averaged distance between electrons

quantum effects important if

6 $$\frac{h}{\sqrt{2mE}} > n^{-1/3}$$

For thermal electrons $E = \frac{3}{2} k_B T$

5 $$n^{1/3} > \frac{\sqrt{3 m k_B T}}{h} = \frac{\sqrt{3 \cdot 9.1 \times 10^{-31} kg \cdot 1.38 \times 10^{-23} J/K \cdot 300K}}{6.63 \times 10^{-34} J \cdot S}$$

$$= 16 \times 10^{+7} m^{-1} = 1.6 \times 10^{8} m^{-1}$$

Atm plasma

4 $n^{1/3} = 1.14 \times 10^{7} \ll 1.6 \times 10^{8}$

quantum effects insignificant

4 metals

$n^{1/3} = 3.9 \times 10^{9} \gg 1.6 \times 10^{8}$

quantum effects are important

QM (A4)

(a) 2 $\psi(x) = 0, \; x < 0$

5 $\psi(x) = C_1 \sin kx + C_2 \cos kx, \; x > 0$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

From the boundary condition at $x=0$ $C_2 = 0$

4 $\psi(x) = C_1 \sin kx$

(b) 5 no since $-i\hbar \frac{d}{dx} \psi \neq$ number $\cdot \psi(x)$

5 momentum eigenstates are e^{ikx}, e^{-ikx} $k = \frac{p}{\hbar}$

4 $\psi(x) = \frac{C_1}{2i}\left(e^{ikx} - e^{-ikx}\right)$

B1

$$\psi(\vec{r}) = R_{2p}(r)\, Y_{1m}(\theta,\phi)$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \quad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}$$

$$R_{2p}(r) = \frac{r}{\sqrt{24}\, a_0^{5/2}} e^{-r/2a_0}$$

2 (a) $m = 0, \pm 1$

2 (b) $P = (-1)^1 = -1$ odd

5 (c) $\langle \cos\theta \rangle = \int \cos\theta \, |Y_{1m}(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi$

$$= \int \cos\theta \, |Y_{1m}(\theta,\phi)|^2 \, d(\cos\theta) = 0$$

since integrand is an odd function of $\cos\theta$

(d) 2 $m = 0$: max of $\cos^2\theta$ @ $\theta = 0, \pi$

2 $m = \pm 1$: max of $\sin^2\theta$ @ $\theta = \frac{\pi}{2}$

(e) 3 $P_{2p} = r^2 R_{2p}^2 = \frac{r^4}{24 a_0^5} e^{-r/a_0}$

(f) 5 $\langle r \rangle = \frac{1}{24 a_0^5} \int r^5 e^{-r/a_0} dr = \frac{a_0^6}{24 a_0^5} 5! = 5 a_0$

(g) 4 max of P_{2p} $\quad \frac{d}{dr}\left(r^4 e^{-r/a_0}\right) = e^{-r/a_0}\left(-\frac{r^4}{a_0} + 4r^3\right) = 0$

$$r = 4 a_0$$

B2

$$V = \frac{m\omega^2 x^2}{2} - Fx$$

Complete square

$$V = \frac{m\omega^2}{2}\left(x^2 - \frac{2F}{m\omega^2}x + \frac{F^2}{(m\omega^2)^2}\right) - \frac{F^2}{2m\omega^2}$$

4
$$= \frac{m\omega^2}{2}(x - x_0)^2 - \frac{F^2}{2m\omega^2}, \quad x_0 = \frac{F}{m\omega^2}$$

(a) $H = \frac{p^2}{2m} + V$ has the same spectrum as H_0

but shifted down by $\frac{F^2}{2m\omega^2}$

4
$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right) - \frac{F^2}{2m\omega^2} \quad n = 0, 1, \ldots$$

(b) ψ_0 for H is the same as for H_0, but with the position shift x_0

4
$$\psi_0^H(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\frac{\alpha^2(x-x_0)^2}{2}} \qquad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$$

(c) probability amplitude

setup 4
$$A = \int_{-\infty}^{\infty} \psi_0^H(x)\,\psi_0^{H_0}(x)\,dx = \frac{\alpha}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\alpha^2 x^2}{2} - \frac{\alpha^2(x-x_0)^2}{2}\right] dx$$

calculate 5
$$= \frac{\alpha}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\alpha^2\left(x^2 - x x_0 + \frac{x_0^2}{4}\right) - \frac{\alpha^2 x_0^2}{4}\right] dx$$

$$= \frac{\alpha}{\pi^{1/2}} e^{-\frac{\alpha^2 x_0^2}{4}} \int e^{-\alpha^2\left(x - \frac{x_0}{2}\right)^2} dx = e^{-\frac{\alpha^2 x_0^2}{4}} = e^{-\frac{F^2}{4m\hbar\omega^3}}$$

$$P = A^2 = e^{-\frac{F^2}{2m\hbar\omega^3}}$$

4 (d) if $F_1 = 2F$ $\quad P_1 = e^{-\frac{2F^2}{m\hbar\omega^3}} = P^4 = (0.1)^4 = 10^{-4}$

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

B3.

(a) $\psi = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1)$ $\quad \psi_0 = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\alpha^2 x^2/2}, \; \alpha^2 = \frac{m\omega}{\hbar}$

1 $\quad \langle x \rangle = \int_{-\infty}^{\infty} \psi^2(x)\, x\, dx = \frac{1}{2}\int (\psi_0^2 + \psi_1^2 + 2\psi_0\psi_1)\, x\, dx$

The first two integrals $= 0$ because integrand is odd

$$\langle x \rangle = \int \psi_0 \psi_1 x\, dx = \frac{\alpha}{\pi^{1/2}} \cdot 2\alpha \int x^2 e^{-\alpha^2 x^2} dx$$

4 $\quad \psi_1 = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} 2\alpha x e^{-\alpha^2 x^2/2} \quad \Big| \quad = \frac{2\alpha^2 \pi^{1/2}}{\pi^{1/2}\, 2\alpha^3} = \frac{1}{\alpha} = \left(\frac{\hbar}{m\omega}\right)^{1/2}$

3 $\quad \langle p_x \rangle = -i\hbar \int \psi \frac{d\psi}{dx} dx = 0$

since ψ is real and p_x is hermitian

More explicitly $\int_{-\infty}^{\infty} \psi \frac{d\psi}{dx} dx = \psi^2\Big|_{-\infty}^{\infty} - \int \frac{d\psi}{dx}\psi\, dx = 0$ $\quad (I = -I)$

3 $\quad \langle H \rangle = \frac{1}{2}E_0 + \frac{1}{2}E_1 = \frac{1}{2}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{3}{2}\hbar\omega = \hbar\omega$

2 $\quad \langle P \rangle = \frac{1}{2}\int (\psi_0 + \psi_1)(\psi_0 - \psi_1)\, dx = 0 \quad$ or $\langle P \rangle = \frac{1}{2}P_0 + \frac{1}{2}P_1 = 0$

$\psi_0(-x) = \psi_0(x)$ $\quad$ since $\int \psi_0(x)\psi_1(x) = 0$

$\psi_1(-x) = -\psi_1(x)$.

3 $\quad$ (b) x and p_x don't commute with H – not conserved

H and P commute – are conserved

3 $\quad$ (c) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_0 e^{-iE_0 t/\hbar} + \psi_1 e^{-iE_1 t/\hbar}\right]$

$$E_0 = \frac{\hbar\omega}{2}, \quad E_1 = \frac{3}{2}\hbar\omega$$

3 $\quad$ (d) $\langle x \rangle = \frac{1}{2}\int x \left[\psi_0^2 + \psi_1^2 + 2\psi_0\psi_1 \operatorname{Re}\left(e^{i(E_1 - E_0)t/\hbar}\right)\right)$

$$\frac{E_1 - E_0}{\hbar} = \omega$$

B3 continued

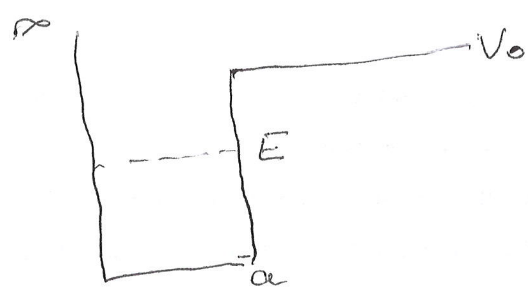
$\langle x \rangle = \cos \omega t \int x \psi_0 \psi_1 \, dx = \left(\frac{\hbar}{m\omega}\right)^{1/2} \cos \omega t$ (from (a))

3 $\langle p_x \rangle = m \frac{d\langle x \rangle}{dt} = \left(\frac{m\hbar}{\omega}\right)^{1/2} (-\omega) \sin \omega t =$

Ehrenfest theorem

$= -(m\hbar\omega)^{1/2} \sin \omega t$

B4.



(a) 1 $x < 0$ $\quad \psi = 0$

4 $0 < x < a$ $\quad \psi_I = A \sin kx$, $\quad k = \frac{\sqrt{2mE}}{\hbar}$

4 $x > a$ $\quad \psi_{II} = B e^{-\lambda x}$ $\quad \lambda = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

(b) match ψ_I and ψ_{II}

4 $A \sin ka = B e^{-\lambda a}$

$kA \cos ka = -\lambda B e^{-\lambda a}$

divide these equations

3 $k \cot ka = -\lambda$ ← equation for eigenvalues E

(c) for $V_0 \to \infty$ $\quad \lambda \to \infty$

3 $\cot ka = \infty$ or $\tan ka = 0$

4 $\longrightarrow$ $ka = \pi n \to E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$

2 $\psi_I = A \sin kx$

$\psi_{II} \to 0$ for $\lambda \to \infty$

corresponds to infinite well