A1.

1. Find the angle θ at which the rope will settle. Express your answers in terms of the given variables M, g, and a as needed.

Solution: Applying Newton's second law to mass M, (1)

+10

$$\begin{aligned} \sum F_y &= 0 = T\cos\theta - Mg \\ \sum F_y &= Ma = T\sin\theta \end{aligned}$$

Solving for θ yields

+5

$$\tan\theta = \frac{a}{g}$$

2. What will the tension T of the rope be once it settles into this angle? Express your answers in terms of the given variables M, g, and a as needed.

Solution: Taking the result for θ obtained in the previous step and substituting into the equations of motion yields

+10

$$T = M\sqrt{g^2 + a^2}$$

From (1) $T = \frac{Mg}{\cos\theta} = Mg\sec\theta = Mg\sqrt{1+\tan^2\theta} = Mg\sqrt{1+a^2/g^2}$
$= M\sqrt{a^2+g^2}$

A2.

Solution: Applying Newton's second law to masses 1 and 2 yields:

+15

$$\begin{aligned} m_1 a_1 &= -m_1\,\omega^2 \cdot d = -T_A + T_B \\ m_2 a_2 &= -m_2\,\omega^2 \cdot (2d) = -T_B \end{aligned}$$

where we have used that the masses have a common angular acceleration, $\alpha = \omega^2$, to express the left-hand side in terms of the angular rotation ω. It follows that

+5

+5

$$\begin{aligned} T_B &= m_2\omega^2 d \\ T_A &= (2m_2 + m_1)\,\omega^2 d \end{aligned}$$

$$\left.\begin{aligned} T_B - T_A &= -m_1\omega^2 d \\ -T_B &= -2m_2\omega^2 d \end{aligned}\right\} \Rightarrow -T_A = -m_1\omega^2 d - 2m_2\omega^2 d$$

$$\Rightarrow T_A = \omega^2 d(m_1 + 2m_2)$$

A3.

Solution:

$+5$

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{x}^2 - U(x) \\ &= \frac{1}{2}m\dot{x}^2 - \frac{k}{x}e^{-t/\tau}, \end{aligned}$$

where the potential $U(x)$ is determined from $F(x,t)$,

$+5$

$$U(x) = -\int F dx = \frac{k}{x}e^{-t/\tau}$$

The Hamiltonian is $H = p_x\dot{x} - L$ with canonical momentum given by

$+5$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

Substituting and simplifying yields

$+5$

$$\begin{aligned} H &= p_x\dot{x} - L \\ &= \frac{p_x^2}{2m} + \frac{k}{x}e^{-t/\tau} \end{aligned}$$

H must be expressed in terms of P NOT $\dot{x}$

The Hamiltonian is the total energy. $\frac{\partial L}{\partial t} \neq 0$, thus energy is not conserved.

$+5$

A4

(1) May Preliminary Exam problems

By Marcel Ngoko

Classical Mechanics

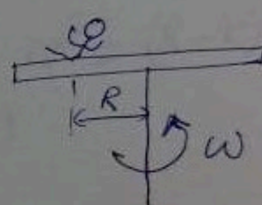
pb #1: Easy –

~~U~~Under the critical circumstance that the astronaut just starts to slide, one has:

$$f_s = m\,a_n$$

where $f_s = \mu_s N = \mu_s m g$ is the static friction

$a_n = \frac{v^2}{R} = R\omega^2$ is the centripetal acceleration

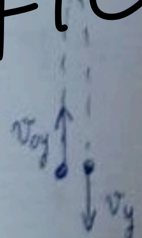


+10

$\Longrightarrow\ mR\omega^2 = \mu_s m g$

$\Longrightarrow\ \boxed{R = \frac{\mu_s g}{\omega^2}}$ where g on that planet has to be determined.

+10



$v_y = v_{oy} + a_y t \Longrightarrow a_y\ \cancel{t} = \frac{v_y - v_{oy}}{t}$

$= \frac{-(30\,m/s)\times 2}{20\,s}$

$\Longrightarrow g = |a_y| = 3.0\,m/s^2$

$R = \frac{0.4 \times \overset{3}{\cancel{1.6}}}{2^2} = 0.30\,m = 30\,cm$ +5

R is independent on m.

B1.

Solution: The position of the stone as a function of time is

+4

$$s(t) = h - \frac{1}{2}gt^2,$$

while the position of the rocket can be determined starting from $a_y(t)$:

+5

+5

$$\begin{aligned} a_y(t) &= A - Bt \\ v_y(t) &= \int_0^t a_y(t')dt' = At - \frac{1}{2}Bt^2 \\ r(t) &= \int_0^t v_y(t')dt' = \frac{1}{2}At^2 - \frac{1}{6}Bt^3 \end{aligned}$$

The maximum height the rocket attains occurs at time T when $v_y(T) = 0$.

+5

$$\begin{aligned} v_y(T) &= 0 = T\left(A - \frac{1}{2}BT\right) \\ T &= 0, \frac{2A}{B} \end{aligned}$$

We want the latter (non-zero) time. The initial height h can be solved by equating $s(t) = r(t)$ at time $t = T$.

+6

$$\begin{aligned} r(T) &= r_{\max} = \frac{2A^3}{3B^2} \\ s(T) &= h - \frac{1}{2}g\left(\frac{2A}{B}\right)^2 \\ h &= \frac{2A^3}{3B^2} + \frac{1}{2}g\left(\frac{2A}{B}\right)^2 \\ &= \frac{2A^2}{B^2}\left(g + \frac{A}{3}\right) \end{aligned}$$

B2.

Solution: For simple projectile motion under constant acceleration and ignoring air resistance, the equations of motions can be solved yielding

+5

$$\begin{aligned} x(t) &= v_0 \cos\theta t \\ y(t) &= v_0 \sin\theta t - \frac{1}{2} g t^2, \end{aligned}$$

where v_0 is the initial speed of the projectile, θ is the angle above the horizontal at which the projectile is fired, and g is the constant acceleration due to gravity near the surface of the Earth. The distance of the projectile from the launch point is

+5

$$\begin{aligned} d^2 &= x^2(t) + y^2(t) \\ &= t^2 \left[v_0^2 - v_0 \sin\theta g t + \left(\frac{gt}{2}\right)^2 \right] \end{aligned}$$

The distance from the origin increases at least until the projectile reaches its maximum height. Thus, $\frac{d^2}{dt^2}(d^2) > 0$ shortly after $t = 0$ and the distance increases until $\frac{d^2}{dt^2}(d^2)$ becomes negative. So, let's find the turning point, i.e. when $\frac{d^2}{dt^2}(d^2) = 0$.

+10

$$\begin{aligned} \frac{d^2}{dt^2}(d^2) &= 2v_0^2 t - 3v_0 \sin\theta g t^2 + g^2 t^3 = 0 \\ &= t\left(2v_0^2 - 3v_0 \sin\theta g t + g^2 t^2\right) \\ t &= 0, \frac{3v_0 \sin\theta}{2g} \pm \frac{v_0}{2g}\sqrt{9\sin^2\theta - 8} \end{aligned}$$

The first solution is trivial so let's focus on the others. These solutions are real only when the term under the square root is non-negative, i.e.

$$\begin{aligned} 9\sin^2\theta - 8 &\geq 0 \\ \sin\theta &\geq \frac{2\sqrt{2}}{3} \end{aligned}$$

Imaginary time is unphysical and thus for

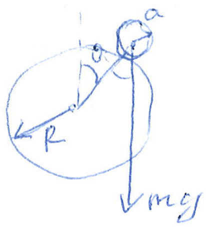
+5

$$\theta < \sin^{-1}\frac{2\sqrt{2}}{3}$$

there is no turning point and $\frac{d^2}{dt^2}(d^2) > 0$.

B3.

B3



(a) for rolling sphere 2nd Newton law

$$\frac{mv^2}{R+a} = mg\cos\theta - N$$

where N is normal reaction

when the sphere goes off $N=0$

+10

$$\frac{mv^2}{R+a} = mg\cos\theta \qquad (1)$$

conservation of energy (no slipping)

$$mg(R+a)(1-\cos\theta) = \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

$$\omega = \frac{v}{a} \text{ (no slipping)} \quad I = \frac{2}{5}ma^2$$

$$mg(R+a)(1-\cos\theta) = \frac{m}{2}\left(v^2 + \frac{2}{5}v^2\right)$$

from (1) $\frac{v^2}{R+a} = g\cos\theta$

+10

$$g(1-\cos\theta) = g\frac{1}{2}\cos\theta \cdot \frac{7}{5}$$

$$\cos\theta = \frac{1}{1+\frac{7}{10}} = \frac{10}{17} \qquad \theta = 54.0°$$

Accept $\theta = \cos^{-1}\left(\frac{10}{17}\right)$ or $\cos\theta = \frac{10}{17}$

(b)

$$L = T - V = \frac{m}{2}\left(v^2 + \frac{2}{5}v^2\right) - mg(R+a)(1-\cos\theta)$$

Choosing θ as a generalized coordinate, $v = R\dot{\theta}$

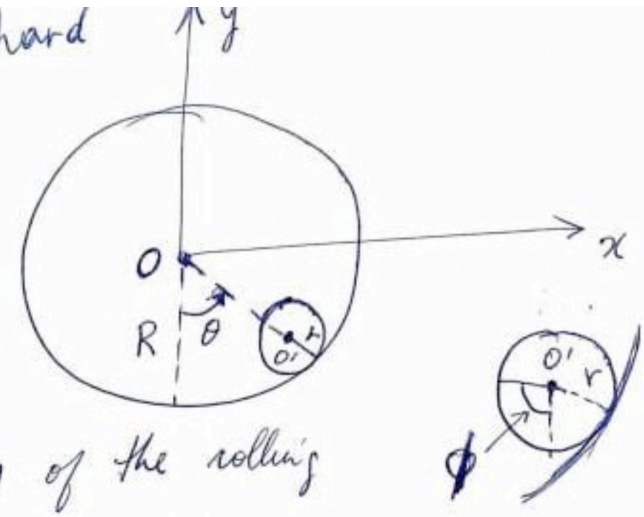
+5

$$L = \frac{7}{10}mR^2\dot{\theta}^2 - mg(R+a)(1-\cos\theta)$$

B4

problem #2 : hard

classical mechanics



A) velocity of the cm of the rolling cylinder

$$\begin{cases} x = (R-r)\sin\theta \\ y = -(R-r)\cos\theta \end{cases} \quad (1)$$

+5 $\Rightarrow$ $\boxed{\begin{cases} \dot{x} = (R-r)\dot{\theta}\cos\theta \\ \dot{y} = (R-r)\dot{\theta}\sin\theta \end{cases}}$ (2) $\quad v^2 = (R-r)^2\dot{\theta}^2$ (3)

B) $L = T - V$

$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\phi}^2$ (4), $I = \frac{1}{2}mr^2$ (5)

$T = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\phi}^2$

$V = +mgy = -mg(R-r)\cos\theta$

+7 $\Rightarrow$ $\boxed{L = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\phi}^2 + mg(R-r)\cos\theta}$ (6)

Also accept $\dot{\phi}$ written in terms of $\dot{\theta}$ using the no-slip condition

2) Condition for rolling without slipping

$$\boxed{(R-r)\theta = r\phi \iff (R-r)\dot{\theta} = r\dot{\phi}} \qquad (7)$$

c) Lagrange's equation of motion wrt θ.

(7) in (6) $\iff$ $L = \frac{1}{2} m (R-r)^2 \dot{\theta}^2 + \frac{1}{4} m \cancel{r^2} \frac{(R-r)^2\dot{\theta}^2}{\cancel{r^2}}$

$$+ mg(R-r)\cos\theta$$

$$L = \frac{3}{4} m (R-r)^2 \dot{\theta}^2 + mg(R-r)\cos\theta \qquad (8)$$

equation of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (9)$

- $\frac{\partial L}{\partial \theta} = -mg(R-r)\sin\theta \qquad (10)$
- $\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2} m (R-r)^2 \dot{\theta} \qquad (11)$

(10) and (11) in (9) lead to:

$$\frac{3}{2} m (R-r)^2 \ddot{\theta} + mg(R-r)\sin\theta = 0$$

$$\Longrightarrow \boxed{\ddot{\theta} + \frac{2}{3}\left(\frac{g}{R-r}\right)\sin\theta = 0} \qquad (12)$$

+8

d) Small oscillations $\theta \ll 1 \Rightarrow \sin\theta \simeq \theta$ (rad)

$$\Rightarrow \boxed{\ddot{\theta} + \frac{2}{3}\left(\frac{g}{R-r}\right)\theta = 0}$$

equation characteristic of simple harmonic

with angular frequency $\omega^2 = \frac{2}{3}\frac{g}{R-r}$ and thus

period:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

+5

$$\boxed{T = \pi\sqrt{\frac{6(R-r)}{g}}}$$

Also accept approximating the Lagrangian for small θ and then finding E-L equations

A1. A 600 g copper ball has a temperature of 700°C when it is placed in 3.00 kg of water at a temperature of 20°C. Calculate the temperature (in °C) of the system when equilibrium has been reached? Assume the system is thermally insulated. Data: $C_{\text{water}} = 4.18\ \text{J g}^{-1}\ \text{K}^{-1}$, $C_{\text{Cu}} = 0.39\ \text{J g}^{-1}\ \text{K}^{-1}$

Answer

Heat entering water = heat leaving copper } 6 points

$$m_w C_w (T_f - T_w) = m_{Cu} C_{Cu} (T_{Cu} - T_f) \Rightarrow$$
$$m_w C_w T_f - m_w C_w T_w = m_{Cu} C_{Cu} T_{Cu} - m_{Cu} C_{Cu} T_f \Rightarrow$$
} 6 points

$$T_f = \frac{m_w C_w T_w + m_{Cu} C_{Cu} T_{Cu}}{m_w C_w + m_{Cu} C_{Cu}} = 305\ \text{K} = 32.5\text{°C}$$
} 13 points

A2. One mole of diatomic ideal gas ($C_V = 2.5\, nR$) performs a transformation from an initial state for which temperature and volume are, $290\ K$ and $30000\ ml$ to a final state in which temperature and volume are 310 K and $16000\ ml$. The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

Solution:

A straight line in (P,V) diagram: $P = P1 + \frac{(V-V1)\Delta P}{\Delta V}$, } 6 points

$W = \int PdV = \int [P1 + \frac{(V-V1)\Delta P}{\Delta V}] dV = [P1 - \frac{V_1 \Delta P}{\Delta V}]\Delta V + \frac{1}{2}\Delta P (V1+V2) = -1690\ \text{J}$ } 7 points

$\Delta U = C_V \Delta T = 416\ J$

$Q = \Delta U + W = -1275\ J$ } 12 points

A3. Consider a gas with the following speed probability distribution $f(v) = A$ when $0 < v < v_0, = 0$ otherwise. Find (a) average speed, (b) rms speed in the 1 dimensional, 2 dimensional, and 3 dimensional cases.

Solution:

$$\text{1D: } <v> = \frac{\int f(v) v dv}{\int f(v) dv} = v_0/2,\ v_{rms}^2 = \frac{\int f(v) v^2 dv}{\int f(v) dv} = v_0^2/3$$
} 8 points

$$\text{2D: } <v> = \frac{\int f(v) v d^2v}{\int f(v) d^2v} = \frac{2}{3} v_0,\ v_{rms}^2 = \frac{\int f(v) v^2 d^2v}{\int f(v) d^2v} = v_0^2/2$$
} 8 points

$$\text{3D: } <v> = \frac{\int f(v) v d^3v}{\int f(v) d^3v} = ¾\, v_0,\ v_{rms}^2 = \frac{\int f(v) v^2 d^3v}{\int f(v) d^3v} = v_0^2 3/5$$
} 8 points

+1 point if all coefficients are correct

A4. An ideal gas (γ= 1.4) expand in an adiabatic process to 10 times of its original volume. If the initial temperature is 0 °C, and the initial pressure 1 atm find the final temperature.

Solution:

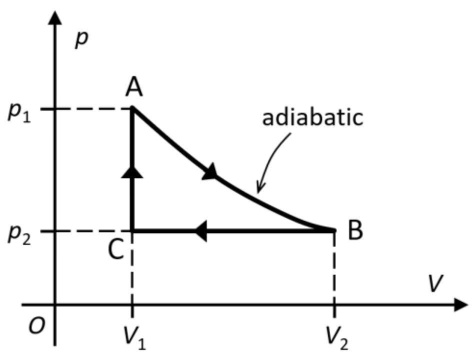
Adiabatic equation 12 points

$T= T_0\ 10^{1-\gamma}$=108 K. $P=P_0/10^{\gamma}$=0.0398 atm

correct solution 13 points

B1. An ideal gas is expanded adiabatically from (p_1,V_1) to (p_2,V_2). It is then compressed isobarically to (p_2,V_1). Finally, the pressure is increased to p_1 at constant volume V_1. Show that the efficiency of the cycle is

$$\eta = 1 - \gamma \frac{V_2 / V_1 - 1}{p_1 / p_2 - 1}$$



Answer

The work the system does in the cycle is

$$W = \oint p dV = \int_{AB} p dV + p_2(V_1 - V_2) \ .$$

10 points

Because AB is adiabatic and an ideal gas has the equations $pV = nkT$ and $C_p = C_v + R$, we get

$$\int_{AB} p dV = -\int_{AB} C_v dT = -C_v(T_2 - T_1)$$
$$= \frac{1}{1-\gamma}(p_2 V_2 - p_1 V_1) \ .$$

10 points

During the CA part of the cycle the gas absorbs heat

$$Q = \int_{CA} T dS = \int_{CA} C_v dT = C_v (T_1 - T_2)$$
$$= \frac{1}{1-\gamma} V_1 (p_2 - p_1) \, .$$

Hence, the efficiency of the engine is

$$\eta = \frac{W}{Q} = 1 - \gamma \frac{\frac{V_2}{V_1} - 1}{\frac{p_1}{p_2} - 1} \, .$$

5 points

B2. The entropy of an ideal gas is $S = n/2\ [a + 5R\ ln(U/n)\ + 2R\ ln(V/n)]$, where n is the mole number, R is the universal gas constant, U is internal energy, V is volume, and a is a constant.

(a) Calculate the constant pressure heat capacity (C_P) and the constant volume heat capacity (C_V).
(b) Rewrite entropy in (T,V), (T,P), and (P,V) representation.

Solution:

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = T\frac{n}{2}\frac{5R}{U}\left(\frac{\partial U}{\partial T}\right)_V = \frac{5}{2}\frac{nRT}{U}\ C_V$$

6 points

Therefore, $U = \frac{5}{2}\ nRT.$

So, $C_V = \frac{5}{2}\ nR$

$$C_P = C_V + nR = \frac{7}{2}\ nR$$

7 points

$$S(T,V) = nc_V \ln \frac{T}{T_o} + nR \ln \frac{V}{V_o} + ns_o.$$

4 points

$$S(T,P) = nc_P \ln \frac{T}{T_o} - nR \ln \frac{P}{P_o} + ns_o.$$

4 points

$$S(P,V) = nc_V \ln \frac{P}{P_o} + nc_P \ln \frac{V}{V_o} + ns_o.$$

4 points

Also accept without P_0, V_0, and T_0.

B3. Consider the adiabatic free expansion of an ideal gas (from volume V to 2V).

(a) What's the work and heat in the process?

(b) How does the temperature change?

(c) Show that this process is irreversible. (Hint: calculate the entropy difference)

(d) answer (b) if the gas is nonideal.

Solution:

6 points

(a) In this process, there is no work and heat. The internal energy does not change, or the temperature does not change.

6 points

(b) Suppose the initial state of the gas can be described as P,V,T, the final state of gas can be described as P/2,2V,T.

7 points

(c) So, one can use an isothermal process (reversible) to bring the system from the initial to the final state, and calculate the entropy change using that process.

Since dU=0,

$$\Delta S = \frac{1}{T} \int (dU + PdV) = \frac{1}{T} \int PdV = \frac{1}{T} \int (\frac{nRT}{V})\, dV = nR \ln (2V/V)$$
$$= nR \ln 2 > 0.$$

Because this process is adiabatic and ΔS>0, it is irreversible.

5 points

(d) If the gas is not ideal, the potential energy of interaction between molecules increases during the expansion, therefore due to conservation of energy the kinetic energy decreases, therefore the temperature slightly decreases

B4.

Thermo

B4.

0: p_0, V_0, T_0
1: $\frac{p_0}{2}, V_1\ T_0$
2: $1.32p_0, V_0, T_2$

8 points

(a) $p_0 V_0 = \frac{p_0}{2} V_1 \rightarrow V_1 = 2V_0$

8 points

(b)

$$\frac{p_0}{2}(2V_0)^{\gamma} = 1.32 p_0 V_0^{\gamma}$$

$$\frac{1}{2} 2^{\gamma} = 1{,}32$$

$$\gamma = \frac{\ln 2.64}{\ln 2} = 1.4$$

for a gas with s degrees of freedom

$$\gamma = \frac{\frac{s}{2}+1}{\frac{s}{2}} = 1 + \frac{2}{s} = 1.4 \rightarrow s = 5$$

$\rightarrow$ gas is diatomic

9 points

(c) look at the temperature

$$TV^{\gamma-1} = \text{const}$$

$$\frac{T_2}{T_0} = \left(\frac{2V_0}{V_0}\right)^{\gamma-1} \rightarrow T_2 = T_0 2^{\gamma-1} = 2^{0.4} T_0 = 1.32 T_0$$

the translational kinetic energy $= \frac{3}{2} k_B T$ increases by 1.32.