9 points

A1] $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$

a) $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$] 3 points

$\bar{E} = a\bar{r}e^{-br}$ – purely radial.

$\therefore \bar{\nabla}\cdot\bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a r e^{-br}\right)$

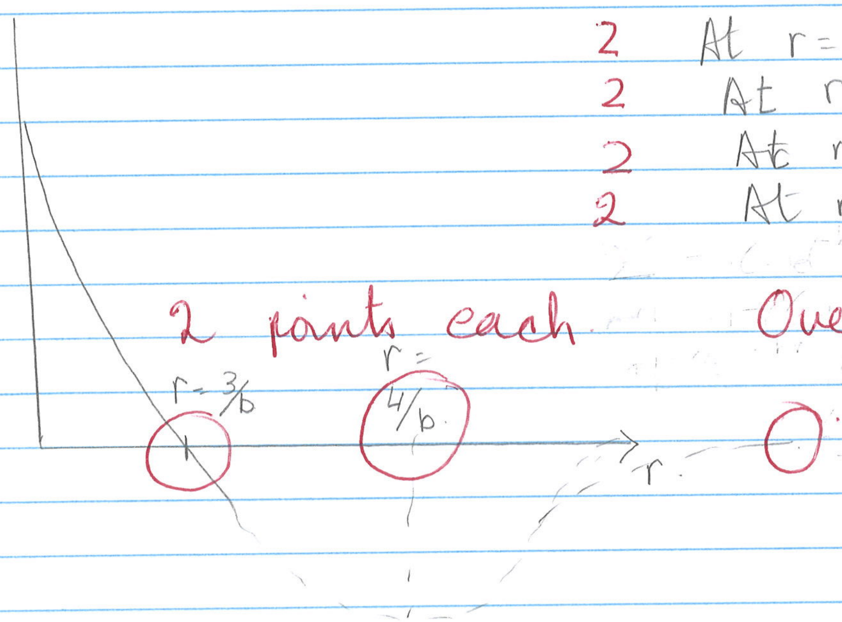
$= \frac{1}{r^2} a \frac{\partial}{\partial r}\left(r^3 e^{-br}\right) = \frac{1}{r^2} a\left[e^{-br} 3r^2 + r^3(-b)e^{-br}\right]$] 3 points

$= 3ae^{-br} - arbe^{-br} = \rho/\epsilon_0$

$\therefore \rho = \epsilon_0 a e^{-br}\left[3 - br\right]$] 3 points.

b

~~8~~10 points



2 At $r = 0$ $\rho = 3a$

2 At $r = 3/b$, $\rho = 0$

2 At $r > 3/b$ $\rho < 0$

2 At $r \to \infty$ $\rho \sim 0$.

2 points each

Overall ~~sh~~ shape 2 points

$\frac{\partial\rho}{\partial r} = \epsilon_0 a e^{-br}\left[-b\right] + \epsilon_0 a(-b)e^{-br}\left[3 - br\right] = 0$

$-\cancel{\epsilon_0}\cancel{a}\cancel{e^{-br}}\cancel{b} - 3\cancel{\epsilon_0}\cancel{a}\cancel{b}\cancel{e^{-br}} + \cancel{\epsilon_0}\cancel{a}b^{\cancel{2}}re^{\cancel{-br}} = 0$

$-1 - 3 + br = 0 \implies br = 4 \implies r = 4/b$

[A1] Continued.

c) What is the total charge of the system?
Answer without integrating ρ.

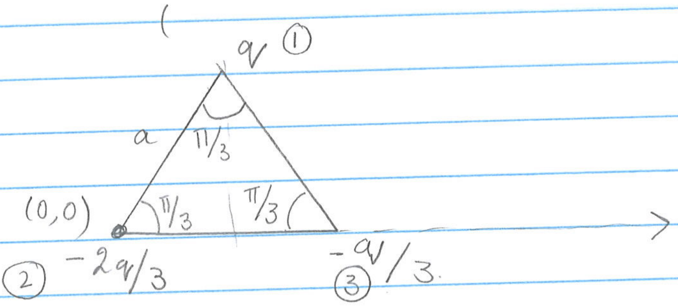
3pts We know $\oint \bar{E} \cdot d\bar{a} = \frac{q_{enc}}{\epsilon_0}$.

Let us integrate E over a sphere of radius r,
$r \longrightarrow \infty$. At $r = \infty$ $E \longrightarrow 0$ $\therefore$ $q_{enc} = 0$

2 points

The graph shows regions of positive AND negative charge, which must cancel out. 1 point.

A 2]



[Dipole moment $\bar{p} = \sum q_i \bar{r}_i$] 2 points

$q_1 = (a/2, a\cos\pi/3)$ q_2 at $(0,0)$ q_3 at $(a,0)$ 6 points

$p_x = q\frac{a}{2} - \frac{qa}{3} = qa\left[\frac{1}{2} - \frac{1}{3}\right] = \left[\frac{3-2}{6}\right] = \frac{qa}{6}$ 6 points

$p_y = q\left[\frac{\sqrt{3}a}{2}\right]$ 4 points.

4 points $\bar{p} = qa\left(\frac{\hat{x}}{6} + \frac{\hat{y}\sqrt{3}}{2}\right)$; $|p| = qa\left[\frac{1}{6^2} + \frac{3}{4}\right]^{1/2}$

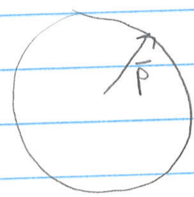
$\tan\Theta = \frac{\sqrt{3}}{2}6 = 3\sqrt{3} \Longrightarrow \Theta$ $= qa\left[\frac{1+27}{36}\right]^{1/2}$

$\Longrightarrow \Theta = 1.38$ rad, $= 79.1°$ $= qa\left[\frac{28}{36}\right]^{1/2}$

$\boxed{|p| = \frac{\cancel{2}qa}{\cancel{6}}\frac{\sqrt{7}}{3}}$

3 points.

A 3]



$\bar{P} = k\bar{r}$

a) Surface + volume bound charges.

Surface: $\sigma_b = \bar{P} \cdot \hat{n} = \bar{P} \cdot \hat{r} = kR \cdot \hat{r} = kR$ 5 points

$\boxed{\sigma = kR}$

Volume: $\rho_b = -\bar{\nabla} \cdot \bar{P}$ only radial dependence

8 points.

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r\right)$$

$$= -\frac{1}{\cancel{r^2}} 3\cancel{r^2} k . \qquad \boxed{\rho_b = -3k}$$

b) Field inside + outside the sphere.

ONLY bound charge (no free charge)

Inside at $r < R$ 6 points

$$\oint \bar{E} \cdot d\bar{a} = \frac{q_{enc}}{\epsilon_0} = \int -3k\, r^2\, 4\pi\, dr$$

$$\therefore E\, \cancel{4\pi}\, \cancel{r^2} = -\frac{\cancel{4\pi}}{\epsilon_0} \cancel{3}k \frac{r^{\cancel{3}}}{\cancel{3}} \qquad E(r) = -\frac{kr}{\epsilon_0}$$

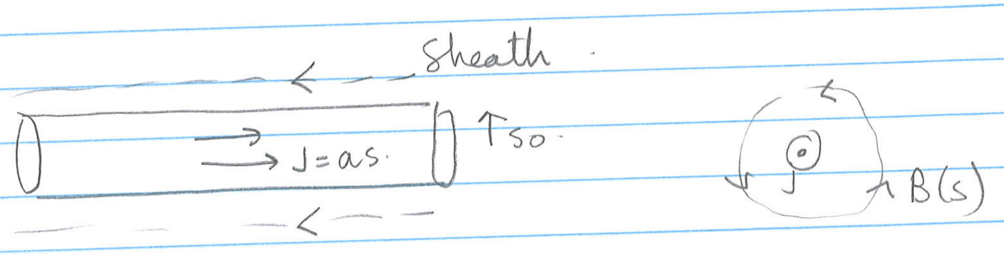
Outside $\oint \bar{E} \cdot d\bar{a} = -\frac{4\pi}{\epsilon_0} \cancel{3}k \frac{R^3}{\cancel{3}} + \frac{kR\, 4\pi R^2}{\epsilon_0}$.

7 points

$$\therefore E\, 4\pi R^2 = -\frac{4\pi k R^3}{\epsilon_0} + \frac{4\pi k R^3}{\epsilon_0} = 0$$ As expected Total charge = 0.

A4]

a)



<u>Inside</u>
Use an amperean loop of radius $s < s_0$.

6 points

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed} \rightarrow 2$$

$$B \cdot 2\pi s = \mu_0 \int \bar{J} \cdot d\bar{a} = \mu_0 a \int s \, 2\pi s \, ds \rightarrow 2$$

4 points

$$\cancel{2\pi}\cancel{s} B = \mu_0 a \cancel{2\pi} \frac{s^{\cancel{3}2}}{3} \qquad \boxed{B = \frac{\mu_0 a s^2}{3} \hat{\phi}}$$

2 points B for $s > s_0 = 0$ (0 current)

b) Usually could use $LI = \Phi$ but finding the area $\perp r$ to B is complicated.

$\therefore$ Use energy

U of a B field $= \frac{1}{2\mu_0} \int B^2 d\tau = \frac{1}{2} L I^2$ 6 points.

$I = \int_0^{s_0} J \cdot da$

We use unit length

5 points

$$\therefore \frac{1}{\cancel{2}\mu_0} \int_0^{s_0} \frac{\mu_0^2 a^2 s^4}{9} 2\pi s \, ds = \frac{1}{\cancel{2}} L I^2$$

$= 2\pi a \frac{s_0^3}{3}$

$$= \frac{1}{} L \frac{4\pi^2 a^2 \cancel{s_0^6}}{\cancel{9}}$$

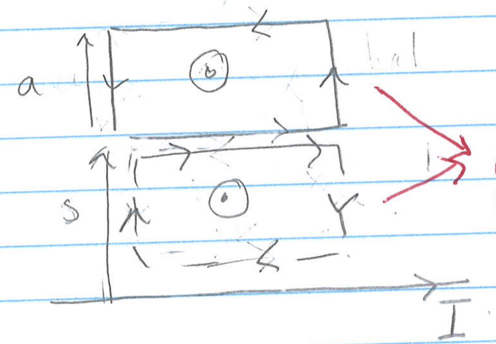
$$\mu_0 a^2 \frac{2\pi}{\cancel{9}} \frac{\cancel{s_0^6}}{6}$$

$$L = \frac{\mu_0 \cancel{a^2} 2\pi}{6 \; 4\pi^2 \cancel{a^2}} = \frac{\mu_0}{12\pi} \text{ per unit length}$$

4 points.

B 1]

a)



~~5~~ 4 points each

The total flux = $\vec{B} \cdot d\vec{a}$

At 90° no flux going through ∴

decreases from initial position

Current goes ccw to increase flux.

As it comes into final position, flux increases

∴ Current goes cw to decrease flux.

b) Total charge

$$\oint \bar{E} \cdot d\bar{l} = -\frac{d\Phi_B}{dt} \quad \text{or} \quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$V\,dt = -d\Phi = -(\Phi_f - \Phi_i) = \Phi_i - \Phi_f$$

4 points

$$\therefore \Delta\Phi = -V\Delta t = -IR\Delta t = -QR$$

$$-\Delta\phi_{tot} = \underbrace{\int_{s_0}^{s_0+a} \frac{\mu_0 I}{2\pi s} a\,ds}_{\Phi_i} - \underbrace{\int_{s_0-a}^{s_0} \frac{\mu_0 I}{2\pi s} a\,ds}_{\Phi_f}$$

$$= \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s_0+a}{s_0}\right) \qquad = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s_0}{s_0-a}\right)$$

~~10~~ ~~8 pts~~ 7 points

[B1] (Cont)

Since the charge flows in OPPOSITE directions for the 2 halves of the rotation

$$\Delta\phi = QR \qquad Q = \frac{\Delta\phi}{R}$$

$$= \frac{\mu_0 I}{2\pi a} \frac{1}{R} \left[\ln\left(\frac{s_0+a}{s_0}\right) \right] - \left(-\ln\left(\frac{s_0}{s_0-a}\right) \right].$$

$$= \frac{\mu_0 I}{2\pi a} \frac{1}{R} \left[\ln\left[\frac{s_0+a}{s_0} \cdot \frac{s_0}{s_0-a}\right]\right]$$

$$= \frac{\mu_0 I}{2\pi a R} \ln\left[\frac{s_0+a}{s_0-a}\right]$$

6 points

B2]

5 points.

16 points.

a) $10\ W/m^2$ at a distance of $r = 1\,m$

$\therefore$ [Over a sphere of $r = 1\,m$, the power is
$4\pi r^2 \times 10 = 40\pi$ Watts $= 125.7$ Watts]

b) The Poynting vector $\bar{S} = \frac{1}{\mu_0}(\bar{E} \times \bar{B})$ is the $\frac{\text{energy}}{\text{unit time area}}$.

At $r = 1\,m$, the energy / unit time / area $= 10\ \frac{W}{m^2}$.

$$10 = \frac{1}{\mu_0} E_0 \cos(kx - \omega t)\, B_0 \cos(kx - \omega t)$$

4 points

We know for an EM wave $B_0 = E_0/c$.

$$\therefore S = \frac{1}{\mu_0} \cdot \frac{E_0^2}{c} \cos^2(kx - \omega t)$$

The time averaging $\frac{1}{T}\int_0^T \cos^2 \omega t\, dt = \frac{1}{2}$.

$$\therefore S = \frac{E_0^2}{2\mu_0 c} \implies E_0 = \sqrt{\frac{2\mu_0 S \times c}{c}} = \sqrt{\frac{2 \times 1.257 \times 10^{-6} \times 10}{3 \times 10^8}}$$

$= 2.7 \times 10$ V/m

$= 27$ V/m

$$B_0 = \frac{E_0}{c} = \frac{2.7 \times 10}{3 \times 10^8} = 0.9 \times 10^{-7}\ T$$

[B2] Continued

[Time averaging $\frac{1}{T}\int_0^T \cos^2(kx-\omega t) = \frac{1}{2}$] 2 points

$\therefore S = \frac{E_0^2}{2\mu_0 c}$

$\therefore E_0 = (2\mu_0 c \times S)^{1/2}$ 2 points

For $r = 1$ m $S = 10$ W/m^2.

$\therefore E_0 = [2 \times 1.257 \times 10^{-6} \times 3 \times 10^8 \times 10]^{1/2}$ 4 points

$E_0 = [75.42 \times 10^2]^{1/2} = 8.68 \times 10$ V/m

87 V/m

$B_0 = \frac{8.68 \times 10}{3 \times 10^8} = 2.89 \times 10^{-7}$ Tesla

At $r = 3$ m Intensity drops by 3^2

$\therefore S = \frac{10}{9}$ W/m^2.

$\therefore E_0 = [2 \times 1.257 \times 10^{-6} \times 3 \times 10^8 \; 10/9]^{1/2}$ 4 points

$\Rightarrow \frac{E(r=1)}{3} = 29$ V/m

$B_0 = 0.96 \times 10^{-7}$ Tesla

4 POINTS

c] Average energy density at $r = 1$ & $r = 3$ m

Energy density $= \frac{\text{Energy}}{\text{Volume}} = \frac{\text{Power} \times \text{time}}{\text{Volume}}$ 2 pts

$= \frac{\text{Energy}}{\text{Area} \times \underbrace{\text{distance/unit time}}_{= c}}$

$= \frac{I}{c}$

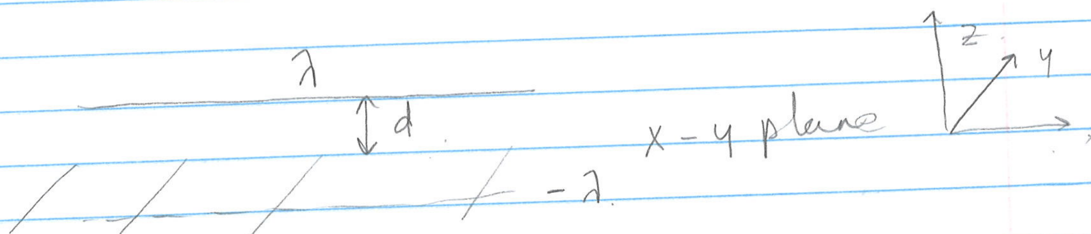
2 points

At 1 m Energy density $= 10/c = 0.33 \times 10^{-7}$ Joules/m^3

At 3 m " " $= \frac{0.33\overset{6}{0} \times 10^{-7}}{9} = 0.0\overset{3}{3}5 \times 10^{-7}$

$= 0.35 \times 10^{-8} \frac{J}{m^3}$

B3]

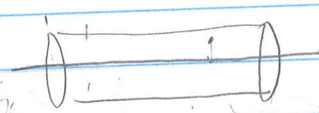


12 points

a) Place an image charge corresponding to a line charge −λ linear charge density at z = −d

Easiest way to get potential is via E-field

3 points



E-field due to a line charge

Gaussian surface is a cylinder. E-field is purely radial

4 points

$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \Longrightarrow E\, 2\pi s \ell = \frac{\lambda \ell}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi s\, \epsilon_0}\hat{s} \qquad \text{where } s = (y^2+(z-d)^2)^{1/2}$$

4 points

$$V(s) = -\int_{s_0}^{s} \frac{\lambda}{2\pi s \epsilon_0}\hat{s}\, ds = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_0}{s}\right)$$

$$V_{\lambda+(-\lambda)} \quad \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{s_0}{s}\right) - \ln\left(\frac{s_0'}{s'}\right)\right] \qquad s_0 = (y^2 +$$

$$V_{TOT} = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{[y^2+(z-d)^2]^{1/2}}{[y^2+(z+d)^2]^{1/2}}\right] =$$

At $z > +d$

[B3] Continued

b) Surface charge density.

$$E_{1n} - E_{2n} = \frac{\sigma_f}{\epsilon_0}$$

$$-\frac{\partial V}{\partial z}\Big|_{z>0} + \frac{\partial V}{\partial z}\Big|_{z<0} = \frac{\sigma_f}{\epsilon_0}$$

3 points.

$$-\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\frac{\partial \ln}{\partial z}\left(\frac{(z-d)^2+y^2}{(z+d)^2+y^2}\right)$$

$$= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\frac{(z+d)^2+y^2}{(z-d)^2+y^2}\left[\frac{1}{(z+d)^2+y^2}2(z-d) - \frac{1}{((z+d)^2+y^2)^2}2(z+d)[(z-d)^2+y^2]\right]\Bigg|_{z=0}$$

$$= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{d^2+y^2}{d^2+y^2}\left[\frac{-2d}{d^2+y^2} - \frac{2d(\cancel{d^2+y^2})}{(d^2+y^2)^{\cancel{2}}}\right]\right]$$

5 points

$$= \frac{\cancel{\epsilon_0}\lambda}{\cancel{4}\pi\cancel{\epsilon_0}}\left[\frac{-4d}{d^2+y^2}\right] = \frac{-d\lambda}{\pi(d^2+y^2)}$$

2 points.

c) For a stripe of width l in x direction + ∞ extent in y direction.

$$Q = -\int_0^l dx \int_{-\infty}^{\infty} dy \frac{d\lambda}{\pi(d^2+y^2)}$$

2 points.

$$= -l\frac{d\lambda}{\pi}\int_{-\infty}^{\infty}\frac{dy}{(d^2+y^2)} = \frac{l\cancel{d}\lambda}{\pi}\frac{1}{\cancel{d}}\,\mathrm{Arctan}\left(\frac{y}{d}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{l\lambda}{\cancel{\pi}}\left[\cancel{\pi}/2 - (-\cancel{\pi}/2)\right]$$

$$\left[Q = -l\lambda\frac{\cancel{2}}{\cancel{2}}\right]$$

2 points

$\boxed{B4}$

a) 1 free electron/atom

~~64 gms $\quad 6.022 \times 10^{23}$ atoms~~

~~9 gms $\quad \dfrac{6.022 \times 10^{23}}{64} \times 9$ atoms~~

~~$n_e = 0.847 \times 10^{23}$ electrons/cc~~

6 points

b) a) $I = n_e e v a$ 3 points

$= 0.847 \times 10^{23} \times 10^{6} \left(\frac{\text{electrons}}{m^3}\right) \times 1.6 \times 10^{-19}$

$\times\, v \times \pi (0.5 \times 10^{-3})^2$

$V_{el} = \dfrac{1 \text{ Amp}}{\pi (0.25) \times 10^{-6} \times 0.847 \times 10^{29} \times 1.6 \times 10^{-19}}$

Check unit $\dfrac{\text{Coul}/s}{m^2 \times \frac{\#}{m^3} \times \text{charge}}$ Coulomb. $= m/s$.

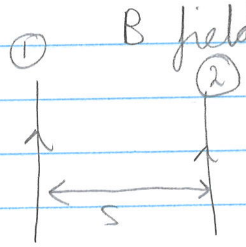
$V_{el} = 0.94 \times 10^{-4}$ m/s

3 points.

8 points

b) Force of attraction

B field at position of wire 2 due to wire 1



$B \cdot 2\pi s = \mu_0 I$

$B = \dfrac{\mu_0 I_1}{2\pi s}$

4 pts.

4 points

$\dfrac{\bar{F}}{\text{unit length}} = q(\bar{v} \times \bar{B}) = \dfrac{\mu_0 I_1 \times I_2}{2\pi s} = \dfrac{\mu_0 I^2}{2\pi \, 1\text{cm}}$

$= \dfrac{1.257 \times 10^{-6} \times 1^2}{2\pi \times 10^{-2}}$

$\vec{F} = 0.2 \times 10^{-4}$ N/meter

B 4] (Cont)

c) Force / unit length

$$\big(\quad \lambda \quad \big) \qquad \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \quad \text{from Gauss Law}$$

where λ = charge/unit length.

3pts If all the positive ions are removed, we know that the # of electrons/volume is 0.847×10^{23}/cc

∴ In a unit length we have

$$0.847 \times 10^{29} \frac{\text{electrons}}{m^3} \times \pi \underbrace{\left[0.5 \times 10^{-3}\right]^2}_{\text{cs area}} \times 1\,m.$$

$$= 0.665 \times 10^{23} \text{ electrons/m}$$

$$\lambda = 1.6 \times 10^{-19} \times 0.665 \times 10^{23} \text{ c/m}$$
$$= 1.064 \times 10^{4} \text{ C/m}.$$

3points Force on 2nd wire $= E \times \lambda = \dfrac{\lambda^2}{2\pi\epsilon_0 s}\hat{s}$

$$\therefore \frac{(1.064)^2 \times 10^8}{2\pi(8.85 \times 10^{-12})\,10^{-2}} = 0.219 \times 10^{8+12+2} = 1.9 \times 10^{20} \text{ N/m}.$$

Magnetic force $= 0.2 \times 10^{-4}$ N/m.

$$\frac{F_{el}}{F_M} = \frac{1.9 \times 10^{20}}{2 \times 10^{-3}} \sim 10^{(+23)} \text{ higher}$$

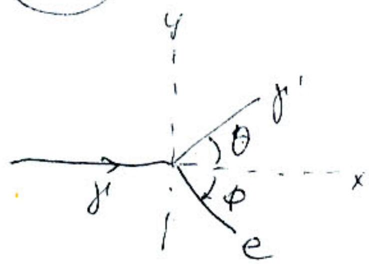
3points Alternatively we can use algebra,

$$\frac{F_{el}}{F_B} = \frac{\lambda^2/\cancel{2\pi}\epsilon_0 \cancel{s}}{\frac{\mu_0}{\cancel{2\pi}} I^2/\cancel{s}} = \frac{1}{\epsilon_0\mu_0}\frac{\lambda^2}{I^2} = c^2\left[\frac{\cancel{n_e} \times \cancel{e} \times \cancel{\text{area}}}{\cancel{n_e} \times \cancel{e} \times v \times \cancel{\text{area}}}\right]^2$$

$$= \frac{c^2}{v^2} = \left[\frac{3 \times 10^8}{0.94 \times 10^{-4}}\right]^2$$

$$1.09 \times 10^{24} \approx 10^{24}$$

A1 QM



Compton formula

$\lambda' - \lambda = \lambda_c(1-\cos\theta)$

$\lambda = \frac{hc}{E} = \frac{1240\,eV\cdot nm}{10^6\,eV} = 1.24\times10^{-3}\,nm$

Since $\theta = \phi$ projection of the photon moment on y axis should be the same as projection of el. momentum (in absolute value)

$\frac{hc}{\lambda'} = p_e$ (1)

6

For projection on x axis $\frac{hc}{\lambda} = \frac{hc}{\lambda'}\cos\theta + p_e\cos\theta$ (2)

6

From (1) and (2)

$\frac{hc}{\lambda} = \frac{2hc}{\lambda'}\cos\theta \rightarrow \lambda' = 2\lambda\cos\theta$

Substitute in compton formula

$2\lambda\cos\theta - \lambda = \lambda_c(1-\cos\theta)$

$\cos\theta = \frac{\lambda_c + \lambda}{\lambda_c + 2\lambda} = \frac{2.43+1.24}{2.43+2.48} = 0.74745$

$\theta = 41.6°$

7

$E' = \frac{hc}{\lambda'} = \frac{hc}{2\lambda\cos\theta} = \frac{E}{2\cos\theta} = 0.669\,MeV$

6

QM

SHORT PROBLEM 1

A2

We consider the operators $T_1 = e^A$ and $T_2 = e^{iA}$, where A is hermitian.

a. Is T_1 hermitian?

b. Is T_2 hermitian?

ANSWER

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a. Yes, T_1 is hermitian:

$$T_1 = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = 1 + A + \frac{1}{2!} AA + \frac{1}{3!} AAA + \dots \quad \Rightarrow$$

$$T_1^{\dagger} = (e^A) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n \right)^{\dagger} = 1^{\dagger} + A^{\dagger} + \frac{1}{2!}(AA)^{\dagger} + \frac{1}{3!}(AAA)^{\dagger} + \dots$$

$$= 1 + A + \frac{1}{2!} AA + \frac{1}{3!} AAA + \dots = e^A = T_1$$

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b. No, T_2 is not hermitian:

$$T_2 = e^{iA} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n A^n = 1 + iA + \frac{1}{2!} i^2 AA + \frac{1}{3!} i^3 AAA + \dots \quad \Rightarrow$$

$$T_2^{\dagger} = (e^{iA})^{\dagger} = \left(\sum_{n=0}^{\infty} \frac{1}{n!} i^n A^n \right)^{\dagger} = 1^{\dagger} + (iA)^{\dagger} + \left(\frac{1}{2!} i^2 AA \right)^{\dagger} + \left(\frac{1}{3!} i^3 AAA \right)^{\dagger} + \dots =$$

$$= 1 + (-i)A + \frac{1}{2!}(-i)^2 AA + \frac{1}{3!}(-i)^3 AAA + \dots = e^{-iA} \neq T_2$$

A3

$$\frac{d}{dt}\langle \varphi(t)|L|\varphi(t)\rangle = \langle \frac{d\varphi}{dt}|L|\varphi(t)\rangle +$$

$$+ \langle \varphi|L|\frac{d\varphi}{dt}\rangle$$ use now Schr. eq.

10

$$i\hbar\left|\frac{\partial\varphi}{\partial t}\right\rangle = H|\varphi\rangle \qquad -i\hbar\langle\frac{\partial\varphi}{\partial t}| = \langle\varphi|H^{+}$$

$H^{+} = H$ (H is hermit.

$$\frac{d}{dt}\langle\varphi|L|\varphi\rangle = -\frac{1}{i\hbar}\langle\varphi|HL|\varphi\rangle + \frac{1}{i\hbar}\langle\varphi|LH|\varphi\rangle =$$ 5

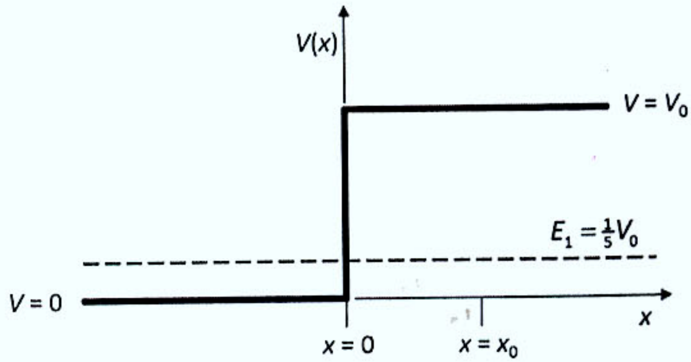
$$= \frac{i}{\hbar}\langle\varphi|HL - LH|\varphi\rangle$$

10

SHORT PROBLEM 2

A4

A particle moves in the potential $V(x)$ shown in the figure. For $x < 0$, the potential is 0. For $x > 0$, it is V_0. The total energy of the particle is $E_1 = \frac{1}{5}V_0$ (dashed line in the figure). Coming from the left, the particle's wavefunction at some position $x = x_0$ (see figure) is $\psi(x = x_0) = \frac{1}{10}\psi(0)$. The total energy is now increased to the value E_2 such that $\psi(x = x_0) = \frac{1}{5}\psi(0)$. Calculate E_2 / V_0.



ANSWER

For total energy $E = E_1$, the evanescent wave is $\psi_1(x_0) = \psi_1(0)e^{-\kappa_1 x_0}$, where $\kappa_1 = \dfrac{\sqrt{2m(V_0 - E_1)}}{\hbar}$, and we have $e^{-\kappa_1 x_0} = 0.1 \;\Rightarrow\; \kappa_1 = \ln(10)/x_0$.

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For total energy $E = E_2$, the evanescent wave is $\psi_2(x_0) = \psi_2(0)e^{-\kappa_2 x_0}$, where $\kappa_2 = \dfrac{\sqrt{2m(V_0 - E_2)}}{\hbar}$, and we have $e^{-\kappa_2 x_0} = 0.2 \;\Rightarrow\; \kappa_2 = \ln(5)/x_0$.

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Hence,

$$\frac{\kappa_2}{\kappa_1} = \frac{\ln(5)}{\ln(10)} = \frac{\sqrt{2m(V_0 - E_2)}/\hbar}{\sqrt{2m(V_0 - E_1)}/\hbar} = \sqrt{\frac{V_0 - E_2}{V_0 - E_1}} = \sqrt{\frac{V_0 - E_2}{\frac{4}{5}V_0}} = \sqrt{\frac{1 - E_2/V_0}{\frac{4}{5}}} \Rightarrow$$

$$E_2 / V_0 = 1 - \frac{4}{5}\left(\frac{\ln(5)}{\ln(10)}\right)^2 = 0.609 \;\Rightarrow\; E_2 = 0.609 V_0$$

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SHORT PROBLEM 3 (61)

The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding (unnormalized) energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

ANSWER

We write the general ket $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$ as a column vector: $|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

The Hamiltonian is then found from

$H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, giving $H = a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. 4

To find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda)-1=0 \Rightarrow -(1-\lambda^2)-1=0 \Rightarrow \lambda = \pm\sqrt{2}$$ 5

so the eigenenergies are $E_{1,2} = \pm\sqrt{2}\,a$

For $\lambda = +\sqrt{2}$:

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow -(1-\sqrt{2})c_1 = c_2 \Rightarrow |\varphi_1\rangle = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$ (unnormalized) 8

For $\lambda = -\sqrt{2}$:

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow -(1+\sqrt{2})c_1 = c_2 \Rightarrow |\varphi_2\rangle = \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$$ (unnormalized) 8

Check using Mathematica:

```
Eigensystem[{{a, a}, {a, -a}}]
{{-√2 a, √2 a}, {{1 - √2, 1}, {1 + √2, 1}}}
```

(QM B2)

QM long 2

A particle with mass m moves in a delta-function potential

$$V(x) = -V_0 a\delta(x)$$

and has total energy $-E<0$. Find the particle's stationary wavefunction and the energy E.

ANSWER

Everywhere but at the origin, the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\psi'' = (-E)\psi \;\Rightarrow\; \psi'' = \frac{2mE}{\hbar^2}\psi = \kappa^2\psi \;\Rightarrow\; \psi(x) = Ae^{\kappa x} + Be^{-\kappa x} \quad \text{with} \quad \kappa = \sqrt{\frac{2mE}{\hbar^2}}$$

We want the wavefunction to be continuous and square-integrable, so we choose

$$\psi(x) = Be^{\kappa x} \qquad \text{for } x<0$$
$$\psi(x) = Be^{-\kappa x} \qquad \text{for } x>0$$

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There is obviously a kink in the wavefunction at $x=0$. To find this kink, we integrate the TISE one time, from $-\varepsilon$ to $+\varepsilon$ (a small interval that include the origin), finding

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\psi'' - V_0 a\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi = -E\int_{-\varepsilon}^{\varepsilon}\psi \;\Rightarrow$$
$$-\frac{\hbar^2}{2m}\left[\psi'(\varepsilon)-\psi'(-\varepsilon)\right] - V_0 a\psi(0) = -2\varepsilon E\psi(0)$$

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If we take the limit $\varepsilon \to 0$, this becomes

$$\psi'(+)-\psi'(-) = -\frac{2mV_0}{\hbar^2}a\psi(0) \;\text{ or }\; -2B\kappa = -\frac{2mV_0}{\hbar^2}aB \;\Rightarrow\; \kappa = \frac{mV_0}{\hbar^2}a$$

Normalization gives

$$\int_{-\infty}^{0}\psi^2 + \int_{0}^{\infty}\psi^2 = 2\int_{0}^{\infty}\psi^2 = 2B^2\int_{0}^{\infty}e^{-2\kappa x}dx = 2B^2\left[\frac{1}{-2\kappa}e^{-2\kappa x}\right]_0^{\infty} = 2B^2\left[\frac{1}{2\kappa}e^{-2\kappa x}\right]_{\infty}^{0} = \frac{B^2}{\kappa} = 1, \text{ so } B = \sqrt{\kappa}$$

The wave function is thus

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$$\psi(x>0) = \sqrt{\kappa}\,e^{-\kappa x}$$
$$\psi(x<0) = \sqrt{\kappa}\,e^{+\kappa x}$$

The energy follows from $\kappa = \sqrt{\frac{2mE}{\hbar^2}} = \frac{mV_0}{\hbar^2}a \;\Rightarrow\; \frac{2mE}{\hbar^2} = \frac{m^2V_0^{\,2}}{\hbar^4}a^2 \;\Rightarrow\; E = \frac{mV_0^{\,2}}{2\hbar^2}a^2$

Name:____

(30 points) **5.** A system consists of two linearly independent states $|1> = \begin{pmatrix}1\\0\end{pmatrix}$ and $|2> = \cancel{\begin{pmatrix}0\\2\end{pmatrix}}$. $\begin{pmatrix}0\\1\end{pmatrix}$

The Hamiltonian has the form $H = \begin{pmatrix}h & k\\k & h\end{pmatrix}$ where h and k are real constants. If the system is initially prepared in state $|2>$ at time $t = 0$, what is its state at a later time t?

Find Eigenvectors and Eigenvalues so that we can write $|2\rangle$ as a linear combination of eigenstates (stationary states) and use the eigenvalues (energy) to add the time dependence.

<u>Eigenvalues</u>: $\quad H|\psi\rangle = E|\psi\rangle \Rightarrow (H - IE)|\psi\rangle = 0$

$$I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\det\begin{pmatrix}h-E & k\\k & h-E\end{pmatrix} = 0 = (h-E)^2 - k^2$$

$$h - E = k \Rightarrow \underline{E_1 = k + h}$$

$$h - E = -k \Rightarrow \underline{E_2 = h - k}$$

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<u>Eigenvectors</u>

$$H|\psi_1\rangle = E_1|\psi_1\rangle$$

$$\begin{pmatrix}h & k\\k & h\end{pmatrix}(c_1|1\rangle + c_2|2\rangle) = \begin{pmatrix}h & k\\k & h\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = (h+k)\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$c_1 h + c_2 k = c_1 h + c_1 k \qquad c_2 = c_1$$
$$c_1 k + c_2 h = c_2 h + c_2 k$$

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$\hookrightarrow$

Name:____

$H|\Psi_2\rangle = E_2 \Psi$

$$\begin{pmatrix} h & k \\ k & h \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (h-k)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\left.\begin{array}{l} c_1 h + c_2 k = c_1 h - c_1 k \\ c_1 k + c_2 h = c_2 h - c_2 k \end{array}\right\} \quad c_2 = -c_1$$

$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (normalized)

$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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Initial state at $t=0$ $\Psi_0 = |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Psi_0 = \left(|\Psi_1\rangle - |\Psi_2\rangle\right) 1/\sqrt{2}$

$$\boxed{\begin{aligned} |\Psi(t)\rangle &= \left(|\Psi_1\rangle e^{-iE_1 t/\hbar} - |\Psi_2\rangle e^{-iE_2 t/\hbar}\right) 1/\sqrt{2} \\ &= \frac{1}{2}\begin{pmatrix} e^{-i(h+k)t/\hbar} & - e^{-i(h-k)t/\hbar} \\ e^{-i(h+k)t/\hbar} & + e^{-i(h-k)t/\hbar} \end{pmatrix} \end{aligned}}$$

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Name:________________________________

2. Consider a particle in a state given by the wavefunction $\Psi(x,t) = Ae^{-x^2/a^2}e^{-iEt/\hbar}$, where A is a real number.

(16 points) a) Find $\langle x \rangle$ and $\langle x^2 \rangle$.

(9 points) b) Assuming this is a state of minimum uncertainty, find the standard deviation of the momentum distribution σ_p.

a) $\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = |A|^2 \int_{-\infty} x e^{-2x^2/a^2} dx = 0$

odd function

$\langle x^2 \rangle = |A|^2 \int_{-\infty}^{\infty} x^2 e^{-2x^2/a^2} dx$ | use $\int_{-\infty}^{\infty} x^2 e^{-bx^2} dx = \frac{\sqrt{\pi}}{2b^{3/2}}$

$b = \frac{2}{a^2}$

$= |A|^2 \frac{\sqrt{\pi}}{2\left(\frac{2}{a^2}\right)^{3/2}} = \frac{|A|^2 \sqrt{\pi}\, a^3}{2 \times 2^{3/2}}$

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Find A from normalization:

use $\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$

$\int |\Psi|^2 dx = 1 = |A|^2 \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx$

$b = 2/a^2$

$1 = |A|^2 \sqrt{\frac{\pi}{2/a^2}} = \sqrt{\frac{\pi}{2}}\, a \Rightarrow |A|^2 = \frac{1}{a}\sqrt{\frac{2}{\pi}}$

$\langle x^2 \rangle = \frac{1}{\cancel{a}} \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}\, a^{\cancel{3}\,2}}{2^{5/2}} = \boxed{\frac{a^2}{4}}$

8

b) $\sigma_x \sigma_p = \hbar/2$ $\quad \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a/2$

$\sigma_p = \frac{\hbar}{2\sigma_x} = \frac{\hbar}{2\, a/2} = \boxed{\frac{\hbar}{a}}$

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