A1. Thermo

$$V = V_0 + ap + bp^2$$ where $a = -0.715\times10^{-3}\ \frac{cm^3}{mole\cdot atm}$

$b = 0.046\times10^{-6}\ \frac{cm^3}{mole\cdot atm^2}$

$$dV = a\,dp + 2bp\,dp$$

$$\int_{p_1}^{p_2} p\,dV = \int_{p_1}^{p_2} p\,(a+2bp)\,dp = a\,\frac{p_2^2-p_1^2}{2} + \frac{2}{3}b\,(p_2^3-p_1^3)$$

$$= -0.715\times10^{-3}\ \frac{1.01^2\,(10^6-1)}{2} + \frac{2}{3}\,0.046\times10^{-6}\cdot1.01^3(10^9-1)\ (cm^3\cdot atm)$$

$$= -0.333\times10^3\ cm^3\cdot atm = -0.333\times10^3\times10^{-6}\times1.01\times10^5 J = -33.3 J$$

This is work done by water

work by environment $= +33.3 J$

Thermo

A4.

(a) The loss of the entropy in the hot reservoir is $\frac{Q}{T_h}$,

$T_h = 500K$

The gain of the entropy in the cold reservoir is $\frac{Q}{T_c}$, $T_c = 300K$

$$\Delta S = \frac{Q}{T_c} - \frac{Q}{T_h} = Q\left(\frac{1}{T_c} - \frac{1}{T_h}\right) = 500\left(\frac{1}{300} - \frac{1}{500}\right) = 0.667 \frac{J}{K}$$

(b) Carnot efficiency $= 1 - \frac{T_c}{T_h}$

$$W = Q\left(1 - \frac{T_c}{T_h}\right) = T_c \Delta S = 200\,J$$

B1. Thermo

$$\Delta Q = p\Delta V + c_V \Delta T$$

(a)

(i) $p = \text{const}$ $\quad W = p\Delta V = p\left(\frac{RT_2}{p} - \frac{RT_1}{p}\right) = R\Delta T$

$$\Delta U = \frac{5}{2} R\Delta T$$

$$\Delta Q = \frac{7}{2} R\Delta T$$

$$\frac{W}{\Delta Q} = \frac{2}{7}$$

(ii) $T = \text{const}$ $\quad W = RT \ln \frac{V_2}{V_1}$

$\Delta U = 0 \qquad \frac{W}{\Delta Q} = 1$

(b) (ii) The process is reversible

$$\Delta S = \int \frac{dQ}{T} = \frac{1}{T}\int p\,dV = \cancel{R} \frac{1}{T}\int_{V_1}^{V_2} \frac{RT}{V}\,dV = R \ln \frac{V_2}{V_1}$$

(i) The process is irreversible, but we can make it quasistatic, so still

$$\Delta S = \int \frac{dQ}{T} = \int_{T_1}^{T_2} \frac{7}{2} R \frac{dT}{T} = \frac{7}{2} R \ln \frac{T_2}{T_1}$$

Thermo

B4.

The probability is given by binomial distribution

$$W(n_1, n_2) = \frac{N!}{n_1! \, n_2!} p^{n_1} q^{n_2} \qquad N = n_1 + n_2$$

where n_1 the number of hops to the right
n_2 to the left, and p, q are corresponding probabilities

position is given by $m = n_1 - n_2$

(a) $N = 10, \; m = 2 \qquad n_1 = \frac{1}{2}(N+m) = 6, \; n_2 = 4 \;, \; p = q = \frac{1}{2}$

$$W(6,4) = \frac{10!}{6!\,4!} \left(\frac{1}{2}\right)^{10} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 4 \cdot 2^{10}} = 0.205$$

$m = 4 \qquad n_1 = 7, \; n_2 = 3$

$$W(7,3) = \frac{10!}{7!\,3!} \left(\frac{1}{2}\right)^{10} = \frac{120}{2^{10}} = 0.117$$

(b) $\bar{n}_{right} = N \cdot p = 5 \qquad \bar{n}_{left} = 5$

(c) $W(6,4) = 210 \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)^4 = 0.088 \qquad p = \frac{4}{5}, \; q = \frac{1}{5}$

$W(7,3) = 120 \left(\frac{4}{5}\right)^7 \left(\frac{1}{5}\right)^3 = 0.201$

$\bar{n}_{right} = Np = 8$

$\bar{n}_{left} = Nq = 2$

A2. Solution:
For ideal gas, V=nRT/P.
So, $\left(\frac{\partial V}{\partial T}\right)_P = \frac{nR}{P}$, $\left(\frac{\partial V}{\partial P}\right)_T = -\frac{nRT}{P^2}$.

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{nR}{PV} = \frac{1}{T},\ K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{V}\ \left(-\frac{nRT}{P^2}\right) = \frac{1}{P}.$$

A3. In a game, you repeatedly roll a standard die with the numbers 1 through 6 on its faces. If you roll a 6, the game is over. If you roll any other number, you may roll again.

a. What is the probability that the game is still not over after *N* rolls?
b. What is the probability that you roll a 6 in the *N*-th roll (so the game is over then)?
c. What is the average number of rolls a player makes in this game? Hint: $(d/du)u^a = au^{a-1}$

Answers

Part a.
Probability to roll a 1, 2, 3, 4, or 5: $p = \frac{5}{6}$
Probability to roll a 6: $q = 1 - p = \frac{1}{6}$
Probability that game is still not over after *N* rolls is $(\frac{5}{6})^N$

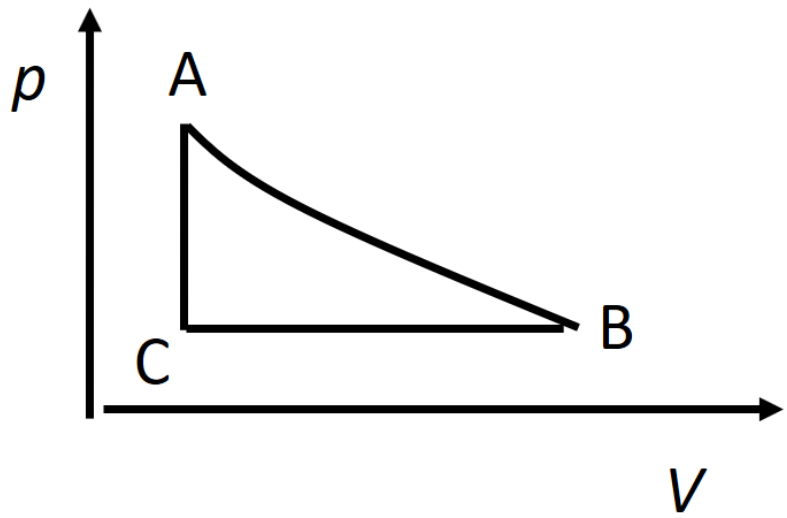
Part b.
Probability that game is not over after *N*–1 rolls is $(\frac{5}{6})^{N-1}$
Probability for a 6 in the *N*-th role is $\frac{1}{6}$ (and then the game is over).
So the answer is $(\frac{5}{6})^{N-1}(\frac{1}{6})$

Part c.

$$\langle N\rangle = \sum_{N=1}^{\infty} Np^{N-1}q = \sum_{N=1}^{\infty} Np^{N-1}(1-p) = (1-p)\sum_{N=1}^{\infty}\frac{d}{dp}p^N = (1-p)\frac{d}{dp}\sum_{N=1}^{\infty}p^N =$$

$$= (1-p)\frac{d}{dp}\left(\frac{1}{1-p}\right) = (1-p)\frac{-1}{(1-p)^2}(-1) = \frac{1-p}{(1-p)^2} = \frac{1}{1-p} = \frac{1}{1-\frac{5}{6}} = 6$$

B2. Solution:



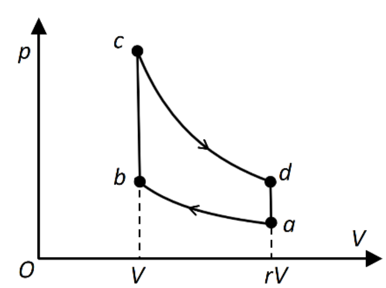
Assume the temperature of A and B is T,
$T_C = P_{BC}V_{AC}/R = \frac{1}{2} P_{BC}V_B/R = \frac{1}{2}$ T.
We then write everything in terms of T.

	ΔU	Q	W
A->B	0	RT ln(2)	RT ln(2)
B->C	-3/4 RT	-5/4 RT	-RT/2
C->A	¾ RT	¾ RT	0

The total work is W=RT [ln(2)-1/2]
The total heat in is Q = RT [ln(2)+3/4]
The efficiency is then W/Q_{in} = 13.4%.

B3.

Consider a one-cylinder Otto-cycle engine with $r = 10.6$. The diameter of the cylinder is 82.5 mm. The distance that the piston moves during the compression is 86.4 mm. The initial pressure (at point *a*) of the gas/air mixture is 8.50×10^4 Pa, and the initial temperature is 300 K (the same as the outside air). Assume that 200 J of heat is added to the cylinder in each cycle by the burning gasoline, and that the gas has $C_V = 20.5$ J/(mol·K) and $\gamma = 1.40$.



a. Calculate the volume of the air-fuel mixture at point *a* in the cycle.

b. Calculate the amount of the mixture in moles.

c. Calculate the temperature of the mixture at points *b*, *c*, and *d* in the cycle.

d. Calculate the efficiency of this engine and compare it with the efficiency of a Carnot-cycle engine operating between the same maximum and minimum temperature.

Solution

(a) The change of the volume

$$\Delta V = A\Delta L = rV - V$$

where $A = \pi D^2/4$ is the area of the base of the cylinder

$$V_b = V = \frac{A\Delta L}{r-1} = \frac{\pi D^2 \Delta L}{4(r-1)} = 4.81 \times 10^{-5}\text{m}^3.$$

$$V_a = rV = 5.10 \times 10^{-4}\text{m}^3.$$

(b)

$$p_a V_a = nRT_a, \quad n = \frac{p_a V_a}{RT_a} = \frac{8.5 \times 10^4 \cdot 5.10 \times 10^{-4}}{8.314 \cdot 300} = 0.01738 \text{ mole}.$$

(c) Point b: For the adiabatic process

$$T_a(rV)^{\gamma-1} = T_b V^{\gamma-1},$$

$$T_b = T_a r^{\gamma-1} = 771 \text{ K},$$

Point c: Heat added

$$Q_H = nC_V(T_c - T_b),$$

therefore

$$T_c = \frac{Q_H}{nC_V} + T_b = \frac{200}{0.01738 \cdot 20.5} + 771 = 1332 \text{ K}.$$

Point d: For the adiabatic process, since $V_c = V$, $V_d = rV$,

$$T_d(rV)^{\gamma-1} = T_c V^{\gamma-1},$$

$T_d = T_c/r^{\gamma-1} = 518$ K.

(d) The rejected heat in $d \to a$

$$|Q_C| = nC_V(T_d - T_a) = 78 \text{ J}.$$

The efficiency

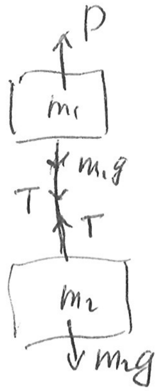
$$e = \frac{Q_H - |Q_C|}{Q_H} = 0.61$$

The Carnot efficiency

$$e_{Carnot} = 1 - \frac{T_C}{T_H} = 1 - 300/1332 = 0.775$$

is larger as should be.

A1.



$$P - T - m_1 g = m_1 a \qquad P_1 = m_1 g \quad P_2 = m_2 g$$

$$T - m_2 g = m_2 a$$

add: $$P - (m_1 + m_2) g = (m_1 + m_2) a$$

$$a = \frac{P - (m_1 + m_2) g}{m_1 + m_2} = g \frac{P - (P_1 + P_2)}{P_1 + P_2}$$

$$T = m_2 (g + a) = m_2 \left(g + \frac{P - (m_1 + m_2) g}{m_1 + m_2} \right) = \frac{m_2 P}{m_1 + m_2}$$

(a) $$\frac{T}{P} = \frac{m_2}{m_1 + m_2} = \frac{P_2}{P_1 + P_2} = 0.8$$

$$a = g \, \frac{150 - 125}{125} = 0.2 g$$

A2.

The energy loss is $-mgh$

is equal to the energy loss due to Friction

$$-mgh = -\frac{\Delta E}{L}x \quad \text{where } x \text{ is the distance travelled}$$

$$x = L\frac{mgh}{\Delta E} = \frac{2.407}{0.688}L = 3.50L$$

So it transfers the flat region 3.5 times, therefore it travels to the left when stops.

$$\cancel{\mu} \quad \Delta E = FL = \mu mgL$$

$$\mu = \frac{\Delta E}{mgL} = \frac{0.688}{2.407} = 0.286$$

A3

$$x(t) = A\cos\omega t$$

$$y(t) = -A\omega \sin\omega t$$

for $t = t_1$ $\quad x(t_1) = \frac{1}{2}A = A\cos\omega t_1 \rightarrow \omega t_1 = \pm\frac{\pi}{3}$

$$v(t_1) = 0.3 = -A\omega \sin\omega t_1$$

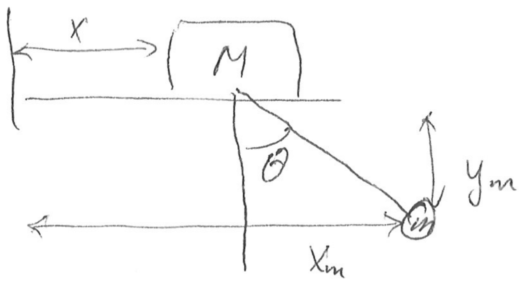
$$0.3 = \mp A\omega \sin\frac{\pi}{3} = \mp A\omega \frac{\sqrt{3}}{2}, \quad A = 0.1\,m$$

$$\omega = \frac{2}{\sqrt{3}} \frac{0.3}{0.1} = 2\sqrt{3}\ rad/s$$

$$m = \frac{k}{\omega^2} = \frac{6\,N/m}{12\,s^{-2}} = 0.5\,kg$$

$$T = \frac{2\pi}{\omega} = \frac{\pi}{\sqrt{3}}\,s$$

A4.



Coordinates of the bob:

$$X_m = x + l\sin\theta \quad \text{therefore} \quad \dot{x}_m = \dot{x} + l\dot{\theta}\cos\theta$$

$$y_m = -l\cos\theta \qquad \dot{y}_m = l\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{\theta}\dot{x}$$

$$U = -mgl\cos\theta$$

$$L = T - U$$

B1. (a) From Bernoulli's law

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_3 + \rho g y_3 + \frac{1}{2}\rho v_3^2$$

$p_1 = p_3 = p_{atm}$, $v_1 = 0$, therefore

$$v_3^2 = 2g(y_1 - y_3) = 2 \cdot 9.8 \cdot 8$$

$$v_3 = 12.5 \frac{m}{s}$$

$A_3 v_3 = A_2 v_2$ $\qquad 0.048 \cdot v_2 = 0.016 \cdot 12.5$

$$v_2 = \frac{12.5}{3} = 4.17 \frac{m}{s}$$

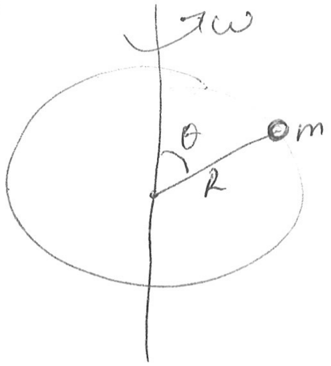
(b) rate $= A_3 v_3 = 0.016 \cdot 12.5 = 0.2 \frac{m^3}{s}$

(c) $$p_2 + \frac{1}{2}\rho v_2^2 = p_3 + \frac{1}{2}\rho v_3^2$$

$$p_2 = p_3 + \frac{1}{2}\rho(v_3^2 - v_2^2) = 1.013 \times 10^5 + \frac{1}{2} \times 10^3 \frac{kg}{m^3}(156.25 - 17.36)$$

$$= 17.07 \times 10^4 \, Pa$$

B2



(a) $T = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m (R \sin\theta)^2 \omega^2$

$U = mgR\cos\theta$

$$L = T - U = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right) - mgR\cos\theta$$

$$\frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta}$$

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

(b) equilibrium: $\ddot{\theta} = 0$

$$-\omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

$$\theta_1 = 0,\ \theta_2 = \pi,\ \theta_3 = \cos^{-1}\left(-\frac{g}{R\omega^2}\right)$$

(c) expand the function

$f(\theta) = -\omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta$ near the equilibrium

$f(\theta) = f(\theta_i) - \left[\omega^2\cos 2\theta_i + \frac{g}{R}\cos\theta_i\right](\theta - \theta_i)$ $\quad i = 1, 2, 3,\ f(\theta_i) = 0$

for stable equilibrium we need

$$-\omega^2\cos 2\theta_i - \frac{g}{R}\cos\theta_i > 0$$

$i=1$: $-\omega^2 - \frac{g}{R} > 0$ impossible, therefore equilibrium is always unstable

(B2), p. 2

$i=2 \qquad -\omega^2 + \frac{g}{R} > 0$

Stable for $\omega^2 < \frac{g}{R}$

$i=3 \qquad \cos\theta_3 = -\frac{g}{R\omega^2} \qquad \cos 2\theta_3 = 2\cos^2\theta_3 - 1 = 2\left(\frac{g}{R\omega^2}\right)^2 - 1$

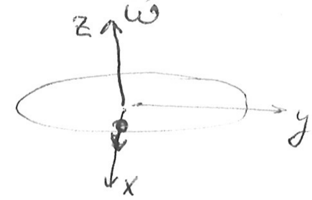
$-\omega^2 \left[2\left(\frac{g}{R\omega^2}\right)^2 - 1\right] + \frac{g}{R}\frac{g}{R\omega^2} > 0$

$-\left(\frac{g}{R\omega}\right)^2 + \omega^2 > 0 \qquad \omega^4 > \left(\frac{g}{R}\right)^2$

Stable for $\omega^2 > \frac{g}{R}$

B3.

The effective force is opposite to ~~for~~ the sum of inertial forces in the rotating frame:



Centrifugal $\vec F_{cf} = -m\vec\omega\times(\vec\omega\times\vec r)$

Coriolis $\vec F_{cor} = -2m\,\vec\omega\times\dot{\vec r}$

transverse force $\vec F_{trans} = -m\dot{\vec\omega}\times\vec r$

$$\vec F_{\cancel{eff}}^{\,inert} = -m\vec\omega\times(\vec\omega\times\vec r) - 2m\vec\omega\times\dot{\vec r} - m\dot{\vec\omega}\times\vec r$$

$$\dot{\vec\omega} = 0 \qquad \vec\omega\times\vec r = \omega r\,\hat z\times\hat x = \omega r\,\hat y$$

$$\vec\omega\times(\vec\omega\times\vec r) = \omega^2 r\,(\hat z\times\hat y) = -\omega^2 r\,\hat x$$

$$\vec F_{cf} = \omega^2 r\,\hat x$$

$$\vec F_{cor} = -2m\vec\omega\times\dot{\vec r} = -2m\omega v\,\hat z\times\hat x = -2m\omega v\,\hat y$$

$$\vec F_{\cancel{eff}}^{\,inert} = m\omega^2 r\,\hat x - 2m\omega v\,\hat y$$

The effective force in the inertial frame $\vec F_{eff} = -\vec F_{inert}$

(b) balance friction with $\vec F_{\cancel{eff}}$ inert

$$\mu m g = \left[(m\omega^2 r)^2 + (2m\omega v)^2\right]^{1/2}$$

$$(\mu g)^2 = (\omega^2 r)^2 + (2\omega v)^2$$

$$r = \frac{\left[(\mu g)^2 - (2\omega v)^2\right]^{1/2}}{\omega}$$

This is the <u>maximum</u> possible value of r before the ant starts to slip

B4

Conservation of energy

$$mgh = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{Iv^2}{2R^2}$$

$$m\left(gh - \frac{v^2}{2}\right) = \frac{Iv^2}{2R^2}$$

$$I = \frac{2mR^2}{v^2}\left(gh - \frac{v^2}{2}\right) = mR^2\left(\frac{2gh}{v^2} - 1\right)$$

Find v from kinematics
For constant acceleration

$$h = \frac{at^2}{2} = \frac{v}{2}t \qquad \text{Since } a = \frac{v}{t}$$

$$v = \frac{2h}{t}$$

(a) $$I = mR^2\left(\frac{gt^2}{2h} - 1\right) = 8.2 \cdot 0.35^2 \cdot \left(\frac{9.8 \cdot 16}{2 \cdot 12} - 1\right)$$

$$= 5.56 \ kg \cdot m^2$$

(b) $$a = \frac{2h}{t^2} = \frac{24}{16} = 1.5 \frac{m}{s^2}$$

(c) $$\alpha = \frac{a}{R} = \frac{1.5}{0.35m} = 4.29 \frac{rad}{s^2}$$