CM

B2

$\cancel{y = y - R\theta}$

$\cancel{\ddot{y} = R\ddot{\theta}}$

$\cancel{\ddot{\theta} = -\frac{2x}{mR} = \frac{2}{mR}\frac{1}{3}mg = \frac{2}{3}\frac{g}{R}}$

(a) $mg - T = m\ddot{x}$

$TR = I\ddot{\theta} \qquad R\theta = x$

$\Rightarrow TR = I\frac{\ddot{x}}{R}$

10 pts

$TR = I\frac{\ddot{x}}{R} \qquad T = \frac{I}{R^2}\ddot{x}$

$I = \frac{1}{2}mR^2$

$mg - I\frac{\ddot{x}}{R^2} = m\ddot{x}$

$mg = \frac{1}{2}m\ddot{x} + m\ddot{x} = \frac{3}{2}m\ddot{x}$

$\ddot{x} = \frac{2}{3}g \qquad T = \frac{1}{2}m\cdot\frac{2}{3}g = \frac{1}{3}g$

(b) $L = \frac{m\dot{x}^2}{2} + \frac{I(\dot{x}/R)^2}{2} + mgx \qquad I = \frac{mR^2}{2}$

$V = -mgx$

$L = \frac{m\dot{x}^2}{2} + \frac{m\dot{x}^2}{4} = \frac{3}{4}m\dot{x}^2 + mgx$

$\frac{\partial L}{\partial \dot{x}} = \frac{3}{2}m\dot{x}$

15 pts

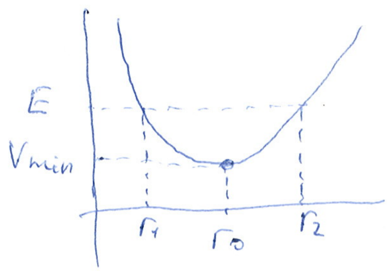
$\frac{3}{2}m\ddot{x} - mg = 0$

$\ddot{x} = \frac{2}{3}g$

CM B3

(a) $V_{eff}(r) = \frac{L^2}{2mr^2} + \frac{kr^2}{2}$

10pts



$$\frac{dV_{eff}}{dr} = 0$$

$$-\frac{L^2}{mr^3} + kr = 0 \qquad r_0^2 = \frac{L}{\sqrt{mk}}$$

$$V_{min} = \frac{L^2\sqrt{mk}}{2mL} + \frac{k}{2}\frac{L}{\sqrt{mk}} = L\sqrt{\frac{k}{m}}$$

For this value of Energy ($E = V_{min}$) r is fixed, therefore the orbit is circular

(b) for $E > E_{min}$ Solve for turning points

5pts

$$E = \frac{L^2}{2mr^2} + \frac{kr^2}{2}$$

$$kr^4 - 2r^2E + \frac{L^2}{m} = 0$$

$$r_{1,2}^2 = \frac{E \mp \sqrt{E^2 - \frac{L^2k}{m}}}{k}$$

r_1 corresponds to the closest approach
r_2 to the farthest distance

(c) orbit is planar because of conservation of the angular momentum $\vec{L}$. Consider the motion in the xy plane
The Lagrangian

10pts

$$\mathcal{L} = T - \frac{k}{2}(x^2+y^2) \qquad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

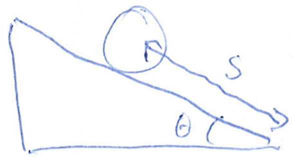
leads to the equations of motion

$$m\ddot{x} + kx = 0, \quad m\ddot{y} + ky = 0$$

Therefore the motion in each coordinate is harmonic with the frequency $\omega = \sqrt{\frac{k}{m}}$

The period $T = 2\pi\sqrt{\frac{m}{k}}$

CM B4



$V = mgs\sin\theta$

$\dot{s}$ is directed upward

(a) $L = \frac{m\dot{s}^2}{2} + \frac{I\dot{\phi}^2}{2} - mgs\sin\theta$

$\dot{\phi} = \frac{\dot{s}}{a} \rightarrow L = \frac{\dot{s}^2}{2}\left(m + \frac{I}{a^2}\right) - mg\,s\sin\theta$

10pts

$= \frac{m\dot{s}^2}{2}\left(1 + \frac{2}{5}\right) - mg\,s\sin\theta = \frac{7}{10}m\dot{s}^2 - mg\,s\sin\theta$

$\frac{d}{dt}\left(\frac{7}{5}m\dot{s}\right) + mg\,s\sin\theta = 0$

5pts (b) $\ddot{s} = -\frac{5}{7}g\sin\theta$

The question says solve Lagrange's equation do we really want a solution written down?

(c) $p_s = \frac{\partial L}{\partial \dot{s}} = \frac{7}{5}m\dot{s} \qquad \dot{s} = \frac{5}{7m}p_s$

5pts $H = \frac{7}{10}m\left(\frac{5}{7m}p_s\right)^2 + mgs\sin\theta = \frac{5}{14m}p_s^2 + mgs\sin\theta$

(d) $\frac{\partial H}{\partial p_s} = \dot{s} \rightarrow \frac{5}{7m}p_s = \dot{s} \rightarrow p_s = \frac{7}{5}m\dot{s}$

5pts $\frac{\partial H}{\partial s} = -\dot{p}_s \rightarrow mg\sin\theta = -\dot{p}_s$

$\ddot{s} = \frac{5}{7}m\dot{p}_s = -\frac{5}{7}g\sin\theta$

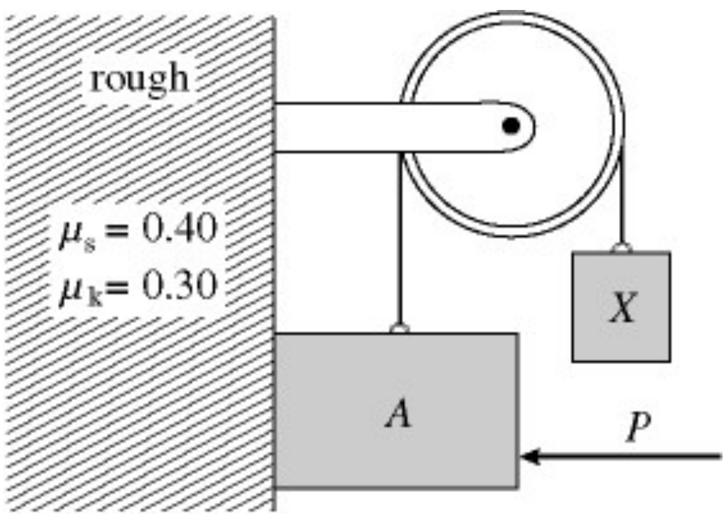
the question says "solve" Do we really want that?

Sample Prelim Questions

July 25, 2022

The first four problem are at the physics 211H level. The following four problems are at a physics 311 level.

Problem 1



Block A of mass 8 kg and block X are attached to a rope that passes over a pulley. A 50 N force P is applied horizontally to block A, keeping it in contact with a rough vertical face. The coefficients of static and kinetic friction between the wall and block A are $\mu_s = 0.4, \mu_k = 0.3$. The pulley is light and frictionless. Determine the mass of block X such that block A descends at a constant velocity of 5 cm/s when it is set into motion.

Solution: Applying Newton's second law to mass A yields

10 pts

$$\begin{aligned} \sum F_x &= 0 = N - P & (1) \\ \sum F_y &= 0 = T + f_k - m_A g, & (2) \end{aligned}$$

where N is the normal force, P is the applied force, T is the tension in the rope, $f_k = \mu_k N$ is the force due to kinetic friction, and g is the acceleration due to gravity at the surface of the Earth. Similarly, for mass X,

5 pts

$$\sum F_y = 0 = T - m_X g. \qquad (3)$$

Obtaining an expression for the tension T from equation (3) and for the normal force N from equation (1) and substituting into equation (2) yields:

10 pts

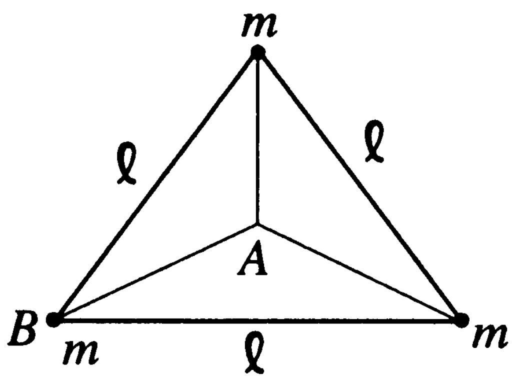
$$\begin{aligned} 0 &= m_X g - m_A g + \mu_k N \\ &= m_X g - m_A g + \mu_k P \\ m_X &= m_A - \frac{\mu_k P}{g} \\ &\sim 6.5 \text{ kg} \end{aligned}$$

Using $g = 10$, $m_X = 8 - \frac{15}{10} = 6.5$

Using $g = 9.8$ $m_x = 8 - 1.5306\cdots$
$= 6.469\cdots$

Accept either value

Problem 2



Three equal masses m are rigidly connected to each other by massless rods of length ℓ forming an equilateral triangle, as shown in the figure above. The assembly is given an angular velocity ω about an axis perpendicular to the triangle. For fixed ω, determine the ratio of the kinetic energy of the assembly for an axis through B compared with that for an axis through A.

Solution: The triangle formed by the massless rods connecting the masses m is an equilateral triangle of length ℓ. Thus, the distance from any mass m to the center of the triangle (which is also the center-of-mass of the system) is ell_{AB}

$$\ell_{AB} = \frac{\ell/2}{\cos 30^\circ} = \frac{\ell}{\sqrt{3}}$$

The ratio of kinetic energies is

10 pts

$$\frac{K_B}{K_A} = \frac{\frac{1}{2} I_B \omega^2}{\frac{1}{2} I_A \omega^2} = \frac{I_B}{I_A},$$

where I_A, I_B are the moments of inertia about an axis perpendicular to the plane in which the masses lie and passing through points A, B respectively. The moment of inertia through point A is $I_A = m \left(\frac{\ell}{\sqrt{3}}\right)^2 \cdot 3 = m\ell^2$. The moment of inertia through point B is simple $I_B = m\ell^2 \cdot 2 + 0 = 2m\ell^2$. Thus,

12 pts for I_A & I_B
2 for final answer

$$\frac{K_B}{K_A} = \frac{I_B}{I_A} = 2.$$

Problem 3

Two cars start 200 m apart and drive toward each other at 10 m/s. A grasshopper jumps back and forth between the cars with a constant horizontal speed of 15 m/s relative to the ground. The grasshopper jumps the instant he lands, so he spends no time resting on either car. What distance does the grasshopper travel before the cars collide?

Solution: The cars collide after $\Delta t = \frac{d}{v_R - v_L} = \frac{200\text{ m}}{20\text{ m/s}} = 10$ s. In this interval, the grasshopper travels a distance $d_g = v_g \cdot \Delta t = 150$ m.

15 pts for set up and realizing only the time matters for the grasshopper distance

10 pts Answer.

Problem 4

A particle of mass 1 kg undergoes one-dimensional motion such that its velocity varies according to $v(x) = \beta x^{-n}$, where β and n are constants and x is the position of the particle as a function of x. Determine the acceleration of the particle as a function of its position x.

Solution:

$$\begin{aligned} v(x) &= \beta x^{-n} \\ a(x) &= \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} \\ &= -n\beta x^{-n-1} \cdot v(x) \\ &= -n\beta^2 x^{-2n-1} \end{aligned}$$

5 pts $a = \frac{dv}{dt}$

10 pts $a = \frac{dv}{dx}\frac{dx}{dt}$

10 pts for answer

B1

A particle of mass m undergoes one-dimensional motion in a potential $U(r) = U_0\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right)$ where r is the distance from the origin, $0 \le r \le \infty$. The quantities U_0, R, λ are positive constants. Find the equilibrium position r_0. For small displacements x from this equilibrium point, show that the potential is quadratic in x. Find the frequency of small oscillations.

Solution: The minimum of the potential occurs at position r_0 that satisfies $\frac{\partial U}{\partial r}|_{r_0} = 0$. Solving yields

10 pts

$$\begin{aligned} U(r) &= U_0\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right), \quad 0 \le r \le \infty \\ \frac{\partial U}{\partial r}|_{r_0} &= 0 = U_0\left(\frac{1}{R} - \lambda^2\frac{R}{r^2}\right), \\ r_0 &= \lambda R \end{aligned}$$

Taking the second derivative $\frac{\partial^2 U}{\partial r^2}$ and evaluating at the equilibrium point r_0,

$$\frac{\partial^2 U}{\partial r^2}|_{r_0} = \frac{2U_0\lambda^2 R}{r_0^3} = \frac{2U_0}{\lambda R^2} > 0.$$

Thus, r_0 is a stable equilibrium point. Substituting $r = r_0 + x$ into U(r) and expanding for small x yields

10 pts

$$\begin{aligned} U(r_0 + x) &= U_0\left(\frac{r_0 + x}{R} + \lambda^2\frac{R}{r_0 + x}\right), \\ &= U_0\left[\frac{r_0}{R}\left(1 + \frac{x}{r_0}\right) + \frac{\lambda^2 R}{r_0}\frac{1}{1 + \frac{x}{r_0}}\right], \\ &\sim \lambda U_0\left[\left(1 + \frac{x}{r_0}\right) + \left(1 - \frac{x}{r_0} + \frac{x^2}{r_0^2} - \cdots\right)\right], \\ &= 2\lambda U_0 + \frac{U_0}{\lambda}\left(\frac{x}{R}\right)^2. \end{aligned}$$

For small displacements x about the equilibrium point r_0 the potential is quadratic in x. The frequency of small oscillations about r_0 is $\omega = \sqrt{\frac{2U_0}{m\lambda R^2}}$. — 5 pts

A simpler solution is to just just Taylor expand about r=r0

$$U = U(r_0) + \cancelto{0}{(r-r_0)U'(r_0)} + \frac{1}{2}(r-r_0)^2 U''(r_0) + \cdots$$

$$= 2\lambda U_0 + (r-r_0)^2\frac{U_0}{\lambda R^2}$$

If the student does the expansion then they can find the frequency just from the expansion and do not need to explicitly compute the second derivative. Accept either way of finding the expansion and frequency.

THERMO – A3

Three factories (A, B, and C) manufacture batteries. Factory A produces 20% of the batteries and factory B produces 75% of the batteries. The remaining 5% of the batteries are from factory C. The defective rate for factory A is 1 in 50, the defective rate for factory B is 1 in 20, and the defective rate for factory C is 1 in 100.

Given that a randomly chosen battery is defective, what is the probability that it came from factory C?

SOLUTION

We need to calculate $P(\text{from C} \mid \text{defective}) = \dfrac{P(\text{from C and defective})}{P(\text{defective})}$

We have

$$P(\text{defective}) = P(\text{from A and defective}) + P(\text{from B and defective}) + P(\text{from C and defective})$$

$$= (0.20) \times \frac{1}{50} + (0.75) \times \frac{1}{20} + (0.05) \times \frac{1}{100} = 0.042$$

and

$$P(\text{from C and defective}) = (0.05) \times \frac{1}{100} = 5 \times 10^{-4}$$

Hence,

$$P(\text{from C} \mid \text{defective}) = \frac{P(\text{from C and defective})}{P(\text{defective})} = \frac{5 \times 10^{-4}}{0.042} = 0.0119 \;\text{ or }\; 1.19\%$$

THERMO B2

The equation of state of some material is

$$pV = AT^3 ,$$

where p, V, and T are the pressure, volume, and temperature, respectively, and A is a constant. The internal energy of the material is

$$U = BT^n \ln(V/V_0) + f(T) ,$$

where B, n, and V_0 are all constants, and $f(T)$ only depends on the temperature.

Find B and n.

SOLUTION

From the first law of thermodynamics, we have

$$dS = \frac{dU + pdV}{T} = \left[\frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{p}{T}\right] dV + \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V dT .$$

We substitute in the above the expressions for internal energy U and pressure p and get

$$dS = \frac{BT^{n-1} + AT^2}{V} dV + \left[\frac{f'(T)}{T} + nBT^{n-2} \ln \frac{V}{V_0}\right] dT .$$

From the condition of complete differential, we have

$$\frac{\partial}{\partial T}\left(\frac{BT^{n-1} + AT^2}{V}\right) = \frac{\partial}{\partial V}\left[\frac{f'(T)}{T} + nBT^{n-2} \ln \frac{V}{V_0}\right] ,$$

giving

$$2AT - BT^{n-2} = 0 .$$

Therefore $n = 3, B = 2A$.

Thermo A1

For diatomic gas $C_v = \frac{5}{2} nR = \frac{5}{2} \cdot 8.31 \frac{J}{K} = 20.8\ J/K$

(a) $Q_v = C_v \Delta T = 20.8 \frac{J}{K} \cdot 80K = 1660\ J$

(b) $C_p = \frac{7}{2} nR = 29.1\ J/K$

$Q_p = C_p \Delta T = 29.1 \frac{J}{K} \cdot 80K = 2330\ J$

(c) from the first law

$$Q_p = \Delta U + \overset{\Delta W}{\cancel{C_v \Delta T + p\Delta V}} = C_v \Delta T + \cancel{p}\,\Delta W$$

$$\Delta W = Q_p - C_v \Delta T = 670\ J$$

Thermo A2

(a) $dQ = dU + pdV$

for isothermal expansion $dU = 0$, using $pV = nRT$

and $\Delta S = \int \frac{pdV}{T} = nR \int_{V_0}^{2V_0} \frac{dV}{V} = nR \ln 2$

(b) entropy is the state function, therefore the entropy change of the gas is exactly the same as in part (a): $\Delta S = nR\ln 2$

Thermo (A4)

$$P = nkT$$

the volume occupied by one molecule is

$$l_0^3 \sim 8 \times 10^{-30}\ m^3$$

For this volume to be noticeable we need

$n l_0^3 > \varepsilon$ where ε is sufficiently small number. Let us take $\varepsilon = 0.1$. Then

$$n l_0^3 = \frac{P}{kT} l_0^3 > 0.1$$

$$P > 0.1 \frac{kT}{l_0^3} = 0.1 \frac{1.38 \times 10^{-23}\ J/K \cdot 300K}{8 \times 10^{-30}\ m^3} \simeq 5.2 \times 10^7\ Pa$$

This is about 520 atm

Thermo B1

(a) The max heat transferred from water is

$$Q = c_w m_w \Delta T = 4186 \frac{J}{kg \cdot K} \cdot 1kg \cdot 30K = 1.256 \times 10^5 J$$

it can melt amount of ice ($L_f = 333.5 \frac{kJ}{kg}$)
latent heat

$$m_{ice} = \frac{125.6 \, kJ}{333.5 \frac{kJ}{kg}} = 0.377 \, kg$$

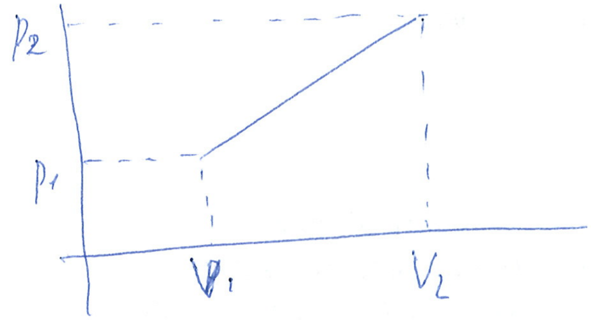
Therefore amount of ice left is $(500-377)g = 123g$

(b) $$\Delta S = \Delta S_w + \Delta S_m = c_w m_w \ln \frac{T}{T_0} + \frac{Q}{T_{ice}}$$

$$\Delta S = 4186 \frac{J}{kg \cdot K} \cdot 1kg \ln \frac{273}{303} + \frac{1.256 \times 10^5 J}{273K}$$

$$= -436.4 + 460.1 = 23.7 \frac{J}{K}$$

Thermo B3.



$$\Delta U = C_v (T_2 - T_1) \qquad C_v = \frac{5}{2} R$$

$$T_1 = \frac{p_1 V_1}{R} \qquad T_2 = \frac{p_2 V_2}{R}$$

$$\Delta U = \frac{5}{2} R \left(\frac{p_2 V_2}{R} - \frac{p_1 V_1}{R} \right) = \frac{5}{2} (p_2 V_2 - p_1 V_1)$$

$$= \frac{5}{2} (2 \times 10^5 \cdot 0.2 - 10^5 \cdot 0.05) = 8.75 \times 10^4 \text{ J}$$

work is given by the area under the line

$$W = (V_2 - V_1) \frac{p_1 + p_2}{2} = (0.2 - 0.05) \frac{3 \times 10^5}{2} = 2.25 \times 10^4 \text{ J}$$

$$Q = \Delta U + W = 1.1 \times 10^5 \text{ J}$$

Thermo (B4)

(a) $p_1 = 10^5\,Pa$ $\quad V_1 = \frac{0.3RT_1}{p_1} = \frac{0.3 \cdot 8.31 \frac{J}{K} \cdot 300K}{10^5 \frac{J}{m^3}} = 7.48 \times 10^{-3} m^3$

$V_2 = V_1 = 7.48 \times 10^{-3} m^3$ $\quad p_2 = \frac{0.3RT_2}{V_2} = 2.00 \times 10^5\,Pa$

$p_3 = p_1 = 10^5\,Pa$ $\quad V_3 = \frac{0.3\,RT_3}{p_3} = 11.34 \times 10^{-3} m^3$

(b) $1 \to 2$: $W_1 = 0$ $\quad \Delta U = 0.3 \cdot \frac{5}{2} R (T_2 - T_1) = \underbrace{0.3 \cdot 2.5 \cdot 8.31}_{6.23} \cdot 300 = 1870\,J$

$Q = \Delta U$

$2 \to 3$ $\quad \Delta U = 0.3 \cdot \frac{5}{2} R (T_3 - T_2) = -6.23 \cdot 145 = -904\,J$

$W_2 = -\Delta U = 904\,J$ $\quad Q = 0$

$3 \to 1$ $\quad \Delta U = 0.3 \cdot \frac{5}{2} R (T_1 - T_3) = -6.23 \cdot 155 = -966\,J$

$W_3 = p_1 (V_1 - V_3) = 10^5 (7.48 - 11.34) \times 10^{-3} = -386\,J$

$Q = -1352$

(c) $W = W_1 + W_2 + W_3 = 518\,J$

(d) $\eta = \frac{W}{Q_{1\to 2}} = \frac{518}{1870} = 0.277$

$\eta_{carno} = 1 - \frac{T_{min}}{T_{max}} = 1 - \frac{300}{600} = 0.5$