UNL - Department of Physics and Astronomy

Preliminary Examination - Day I
Thursday, May 22, 2025

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 “A” questions and 4 “B” questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A

Answer only two Group A questions

A1. For a simple quantum harmonic oscillator with $V(x)=kx^2/2$, show:

(a) $(\Delta p)^2=\langle p^2\rangle$, $(\Delta x)^2=\langle x^2\rangle$, where Δp and Δx are uncertainties in p and x respectively;

(b) the energy of the ground state has the lowest value compatible with the uncertainty principle**.**

A2. (a) Suppose A and B are two scalar Hermitian operators, and [A,B] is their commutator. Show that *i*[A,B] is also Hermitian.

Take now two vector operators **A** and **B.**

(b) Is $i(\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A})$ Hermitian?

(c) Is $i(\mathbf{A}\times\mathbf{B}-\mathbf{B}\times\mathbf{A})$ Hermitian?

A3. A free electron with the kinetic energy 5 eV is radiatively captured by a proton forming a hydrogen atom in the state with the principal quantum number $n=2$.

(a) What is the energy (in eV) and the wavelength (in nm) of the emitted photon?

(b) Find the electron momentum in the initial state. Express it in the SI units and in eV/c where c is the speed of light.

(c) Does the electron have a certain linear momentum in the final state? Explain.

A4. The angular momentum quantum number of the electron in a hydrogen atom is $\ell=5$. What is its lowest possible energy (in eV)? How many nodes between $r=0$ and $r=\infty$ does the corresponding radial wavefunction have?

Quantum Mechanics Group B

Answer only two Group B questions

B1. A particle of mass m is moving in one-dimensional infinite square well:

$V(x)=0$, $|x|<L/2$

$V(x)=\infty$, $|x|>L/2$

(a) Obtain the ground-state wavefunction and the corresponding eigenenergy. Normalize the wavefunction.

(b) Obtain the probability to find the particle in the region $|x|>L/4$.

B2. (a) Find the eigenvalues and eigenstates of the Hamiltonian expressed by the matrix

$$H = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Another observable is represented by a diagonal matrix

$$B = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix}$$

Suppose the system is in an eigenstate of B with the eigenvalue b, and the energy is measured. What are possible outcomes of this measurent, and what are the corresponding probabilities?

B3. We want to study the problem of a particle approaching a one-dimensional asymmetric step potential,

$$V(x) = \begin{cases} 0, & -\infty < x < 0 \\ V_0, & 0 < x < a \\ V_1, & a < x < \infty \end{cases}$$

Assume the energy of the particle to be $E > V_0 > V_1$.

a) Write the general form of the wave function in the three regions defined by the different values of $V(x)$.

b) What are the boundary conditions for the wave function at the points $x = 0$ and $x = a$?

c) Using the fact that the probability current is constant, derive the relation between the transmission and reflection probabilities T and R.

B4. Consider a stationary electron in a uniform magnetic field B pointing along the z direction. At time $t = 0$, the electron is in the eigenstate of spin projection S_y with the spin along the positive y direction..

a. Find the electron state vector at $t=0$
b. Find the electron state vector for $t > 0$.
c. Find the expectation values of S_x for $t > 0$.

Electrodynamics Group A

Answer only two Group A questions

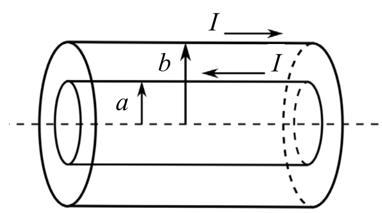
A1. An infinite plane of a uniform surface charge density σ_0 is placed at distance $z = h$ above the surface of a half-space grounded metal.

(1) Find the potential and the electric field in all space.

(2) Find the induced surface charge density on a metal surface.

A2. An electric circuit represents two concentric metal spheres of radii a and b, with $b > a$, serving as electrodes, and a homogeneous material of conductivity σ between the spherical electrodes. Find the resistance R of this material.

A3. Find the self-inductance L per unit length for an infinitely long coaxial cable of radii a, b with $a < b$, carrying a current I, as shown in in the figure.



A4. A beam of protons moves along a straight line with a constant speed through a region 1 of crossed electric and magnetic fields, E=12.4 V/cm and B=0.1 T. It is then directed to a region 2 of a pure magnetic field B=0.005 T

(a) What is the speed of the protons?

(b) What is the radius of the proton's orbit in region 2?

Electrodynamics Group B

Answer only two Group B questions

B1. A slab of a linear dielectric with a uniform dielectric permittivity ε. extends from $z = 0$ to $z=-\infty$. This slab is affected by an external non-uniform electric field whose magnitude and direction in the absence of the dielectric is $\mathbf{E}^{ext}(\mathbf{r})$. There are no free charges in the dielectric.

(1) Show that the normal component of the polarization-induced electric field $E_z^P(\mathbf{r})$ near the surface inside the dielectric is given by $E_z^P(\mathbf{r}) = -\dfrac{\sigma_P(\mathbf{r})}{2\varepsilon_0}$, where $\sigma_P(\mathbf{r})$ is the induced surface polarization charge density

(2) Express $\sigma_P(\mathbf{r})$ is terms of the total electric field at the surface;

(3) Find $\sigma_P(\mathbf{r})$.

B2. An electric field has a wave form $\mathbf{E}(z,t) = E_0 \hat{\mathbf{x}} \cos(kz)\cos(\omega t)$.

(1) Using Maxwell's equations, find the magnetic field $\mathbf{B}(z,t)$;
(2) Find the Poynting's vector **S**;
(3) Find time-averaged Poynting's vector $\langle \mathbf{S} \rangle$;
(4) What conclusion about the wave intensity can be made from the latter result? What kind of wave does the given electric field represent?

B3. Two plane electromagnetic waves propagate in *z* direction and have the form

$$\mathbf{E}_1(\mathbf{r},t) = \hat{\mathbf{x}} E_0 e^{ikz - i\omega t + i\varphi_1},$$

$$\mathbf{E}_2(\mathbf{r},t) = \hat{\mathbf{y}} E_0 e^{ikz - i\omega t + i\varphi_2},$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in *x* and *y* directions respectively and E_0 is a real amplitude.

1. What relationship should φ_1 and φ_2 obey to make the superposition of these two waves, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_2(\mathbf{r},t)$, a linearly polarized wave? What is the angle of the polarization plane of this superposed wave with respect to the *x* axis? Write down the *x*- and *y*-components of the resulting electric field in form of the *real* part of $\mathbf{E}$.
2. What relationship should φ_1 and φ_2 obey to make this wave circularly polarized with right-hand (left-hand helicity? Write down the x- and y-components of the resulting electric field in form of the *real* part of $\mathbf{E}$.

B4. A circular loop of wire with mass *M* carries a current *I* in a uniform magnetic field *B*. It is initially in stable equilibrium with its magnetic moment vector parallel to the magnetic field. The loop is given a small twistabout a diameter and then released.

(a) What is the period of the motion? Ignore the gravity.
(b) Find the time dependence of the induced emf, if the initial angular displacement of the loop is θ_0, and the loop's radius is *r*.

Physical constants

Speed of light .... $c = 2.998 \times 10^{8}$ m/s
Planck's constant .... $h = 6.626 \times 10^{-34}$ J$\cdot$s
Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J$\cdot$s
Electron mass .... $m_e = 9.109 \times 10^{-31}$ kg
Electron's rest energy .... 511.0 keV
Boltzmann constant .... $k_B = 1.381 \times 10^{-23}$ J/K
Compton wavelength ..... $\lambda_C = \frac{h}{m_e c} = 2.426$ pm
Elementary charge .... $e = 1.602 \times 10^{-19}$ C
Proton mass .... $m_p = 1.673 \times 10^{-27}$ kg $= 1836\, m_e$
Atomic mass unit .... 1 u=1.66×10^{-27} kg
Electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m
Bohr radius.................... $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} = 0.5292$ Å
Magnetic permeability.... $\mu_0 = 1.257 \times 10^{-6}$ H/m
Rydberg unit of energy ... $Ry = 13.6$ eV
Rydberg constant……..... R=1.097×10^{7} m^{-1}
1 hartree (= 2 Ry)............. $E_h = \frac{\hbar^2}{m_e a_0^2} = 27.21$ eV
Molar gas constant.......... $R = 8.314$ J / mol$\cdot$K
Gravitational constant...... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2
Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}
hc .... $hc = 1240$ eV$\cdot$nm
Fine structure constant .. $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$
$E^2 = p^2c^2 + m^2c^4$

TRIGONOMETRY

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$\cos(ix) = \cosh(x)$
$\sin(ix) = i\sinh(x)$

For small x:

$\sin x \approx x - \frac{1}{6}x^3$

$\cos x \approx 1 - \frac{1}{2}x^2$

$\tan x \approx x + \frac{1}{3}x^3$

QUANTUM MECHANICS

$\left[AB, C\right] = A\left[B, C\right] + \left[A, C\right]B$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_+ \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell, m+1\rangle$$
$$L_- \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell, m-1\rangle$$

Gyromagneic ratio for electron (SI units) = e/m

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Probability current in one dimension

$$j = \frac{\hbar}{2mi}\left(\psi^*(x)\frac{d\psi(x)}{dx} - \psi(x)\frac{d\psi^*(x)}{dx}\right) = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*(x)\frac{d\psi(x)}{dx}\right)$$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z)=\frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z)=\sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z)=\mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z)=\sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z)=\mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z)=\mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

Stationary states of harmonic oscillator for $n = 0$ and $n = 1$:

$$\varphi_0(x)=\left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2}\mathrm{e}^{-\frac{\alpha^2x^2}{2}},$$

$$\varphi_1(x)=\left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2}2\alpha x\,\mathrm{e}^{-\frac{\alpha^2x^2}{2}},$$

where $\alpha=\left(\frac{m\omega}{\hbar}\right)^{1/2}$.

Hydrogen atom: $E_n=-\frac{Ry}{n^2},\ Ry=\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$

Radial functions for the hydrogen atom $R_{nl}(r)$:

$$R_{10}(r)=\frac{2}{a_0^{3/2}}\exp\left(-\frac{r}{a_0}\right),$$

$$R_{20}(r)=\frac{2}{(2a_0)^{3/2}}\left[1-\frac{r}{2a_0}\right]\exp\left(-\frac{r}{2a_0}\right),$$

$$R_{21}(r)=\frac{r}{24^{1/2}a_0^{5/2}}\exp\left(-\frac{r}{2a_0}\right).$$

$$j=\frac{\hbar}{2mi}\left(\psi^*(x)\frac{d\psi(x)}{dx}-\psi(x)\frac{d\psi^*(x)}{dx}\right)=\frac{\hbar}{m}\mathrm{Im}\left(\psi^*(x)\frac{d\psi(x)}{dx}\right)$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0}\,; \qquad \mathbf{E} = -\nabla\Phi\,; \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = \Phi(\mathbf{r}_1) - \Phi(\mathbf{r}_2)\,; \qquad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,.$$

Work done: $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[\Phi(\mathbf{b}) - \Phi(\mathbf{a})\right]$.

Energy stored in electric field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$.

Multipole expansion: $\Phi(\mathbf{r}) = \dfrac{q}{4\pi\varepsilon_0 r} + \dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{2}\sum_{ij} Q_{ij}\dfrac{x_i x_j}{r^5} + \ldots$

Field of electric dipole:

$$\mathbf{E}(\mathbf{r}) = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{4\pi\epsilon_0 r^3}$$

Monopole moment: $q = \int \rho(\mathbf{r}) d^3\mathbf{r}$.

Dipole moment: $\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r} d^3\mathbf{r}$.

Quadrupole moment : $Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right] d^3\mathbf{r}$ (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$).

Parallel-plate capacitor: $C = \varepsilon_0 \dfrac{A}{d}$.

Spherical capacitor: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$.

Cylindrical capacitor: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L).

Relative permittivity: $\varepsilon_{\text{r}} = 1 + \chi_{\text{e}}$.

Bound charges: $\rho_{\text{b}} = -\nabla\cdot\mathbf{P}$; $\sigma_{\text{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$.

MAGNETOSTATICS

Relative permeability: $\mu_{\text{r}} = 1 + \chi_{\text{m}}$.

Lorentz force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$.

Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$.

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\int \dfrac{Id\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$).

B-field inside of an infinitely long solenoid: $\mathbf{B} = \mu_0 nI\hat{\boldsymbol{\varphi}}$ (n is the number of turns per unit length).

Ampere's law: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl}}$.

Magnetic dipole moment of a planar current distribution: $\mathbf{m} = I \int d\mathbf{a}$.

Force on a magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$.

Torque on a magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$.

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$.

Bound currents: $J_{\text{b}} = \nabla \times \mathbf{M}$; $K_{\text{b}} = \mathbf{M} \times \hat{\mathbf{n}}$.

<u>Maxwell's equations in vacuum</u>

$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	Gauss' law
$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$	Ampere's law with Maxwell's correction

<u>Maxwell's equations in linear, isotropic, and homogeneous media</u>

$\nabla \cdot \mathbf{D} = \rho_{\text{f}}$	Gauss' law
$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J}_{\text{f}} + \frac{\partial \mathbf{D}}{\partial t}$	Ampere's law with Maxwell's correction

Alternative way of writing Faraday's law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}$.

Mutual and self inductance: $\Phi_2 = M_{21} I_1$; $\Phi = LI$.

Energy stored in magnetic field: $W = \tfrac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$.

Wave equations in a conducting medium:

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}; \qquad \nabla^2 \mathbf{B} = \mu\sigma \frac{\partial \mathbf{B}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int_0^\infty \frac{1}{1+bx^2}\,dx = \frac{\pi}{2b^{1/2}}$$
$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$$
$$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}\,dx = \frac{\frac{bx}{x^2+b^2}+\arctan\left(\frac{x}{b}\right)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) =$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$
$$\int x^4 e^{-x}\,dx = -e^{-x}\left(x^4+4x^3+12x^2+24x+24\right)$$
$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$
$$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2a}$$
$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$
$$\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2a^2}$$
$$\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$
$$\int_0^\infty x^5 e^{-x^2}\,dx = \frac{1}{a^3}$$
$$\int_0^\infty x^6 e^{-x^2}\,dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$$