UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Monday, May 17, 2021

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1. Calculate the thermal energy of 1 mol of Cu at T = 340 K using the classical theory.

A2. How much thermal power is generated by a cylindrical wire at a temperature of T=3200 K? The wire is 80 μm in diameter and 5 cm long?

A3. 300 g block of aluminum at 100 °C is placed in a calorimeter cup with 400 g of water. The mass of the copper calorimeter cup is 80 g. The initial temperature of the water and the cup is 22 °C. What's the final temperature? Specific heats of water, aluminum and copper are 4.184, 0.904, and 0.383 J/(g·K) respectively.

A4. In a vacuum tube of pressure 1.333×10^{-3} Pa, at 27 °C, calculate (1) the number of gas particles per m^3, (2) the volume occupied per particle, (3) the average distance between particles.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1. An object with constant heat capacity C is initially at temperature T_1. It is brought into contact with a heat reservoir at temperature T_R, where $T_R < T_1$.

a) Find the entropy change of both the object and the reservoir.

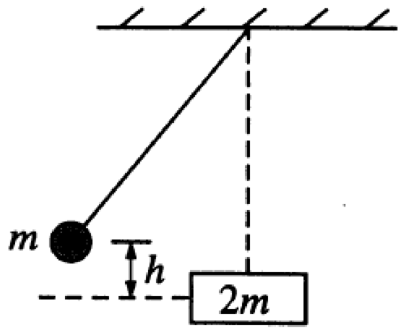
b) Show that the total change in entropy is consistent with the second law of thermodynamics.

B2. One mole of diatomic ideal gas ($C_V = 2.5\, nR$) performs a transformation from an initial state for which temperature and volume are 291 K and 21,000 mL to a final state in which temperature and volume are 305 K and 12,700 mL. The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

B3. Show that, for ideal gas, if the heat capacity C of a process is constant, then the process is described by the equation $PV^a = \text{const}$. Assume that C_v is constant, and find the expression for exponent a in terms of C and C_v.

B4. A diatomic gas ($C_V = 2.5\, nR$) expands adiabatically to a volume 1.35 times larger than the initial volume. The initial temperature is 18 °C. Find the final temperature.

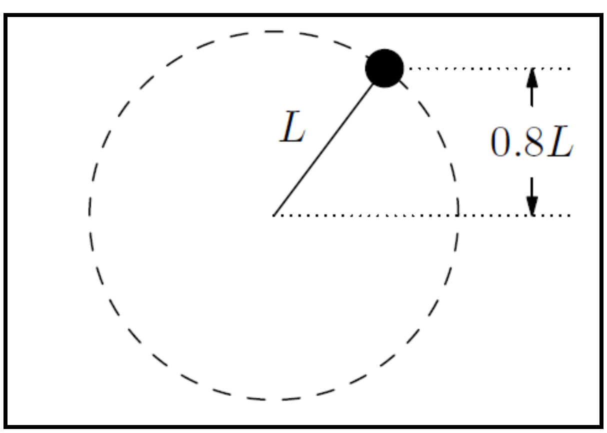
<u>**Classical Mechanics Group A** - Answer only two Group A questions</u>



A1. As shown in the figure above, a ball of mass m, suspended on the end of a wire, is released from a height h. When it is at its lowest point, it collides with a block of mass $2m$ at rest on a frictionless surface. The ball and block stick together. After the collision, to what height does the combined mass rise? Express your result in terms of h.

A2. A circular hoop hangs from a nail on a barn wall. The hoop has a mass m and radius R. If it is displaced slightly by a passing breeze what is the resulting period of small oscillations?

A3. A particle of mass 1 kg undergoes one-dimensional motion such that its velocity varies according to $v(x) = \beta x^{-n}$, where β and n are constants and v is the position of the particle as a function of x. What is the acceleration of the particle as a function of x?



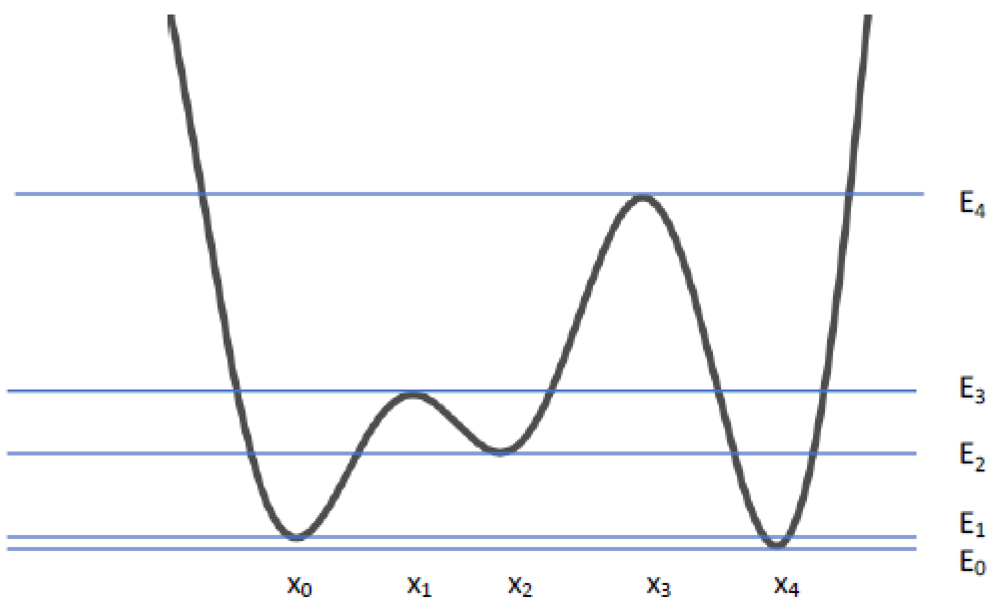
A4 An object of mass m is attached to the end of a massless rod of length L. The other end of the rod is attached to a frictionless pivot. The object is raised so that its height is $0.8L$ above the pivot, as shown in the figure. After the object is released, what is the tension in the rod when it is horizontal?

Classical Mechanics Group B - Answer only two Group B questions

B1. A train moves along a straight horizontal track at a constant speed u. A woman on the train throws a ball of mass m straight ahead with a speed v with respect to herself.

1. What is the kinetic energy gain of the ball as measured by a person on the train?
2. What is the kinetic energy gain of the ball as measured by a person standing by the track?
3. How much work is done by the woman throwing the ball?
4. How much work is done by the train? Explain why this work is nonzero.

B2. A pendulum consists of a mass m suspended by a massless spring with the unextended length b and spring constant k. Find the equations of motion.



B3. Consider a particle of mass m in the potential shown in the figure above.

1. How many equilibrium points are in the figure above? List all stable equilibrium points.
2. Suppose a particle is sitting at point x_2. How much work must be done on the particle for it to visit the region $x < x_1$?
3. Suppose you do work on the particle in accordance with the results you obtained in (2). What is speed of the particle as it passes through the point x_0?
4. Consider the potential energy function $U(x) = a/x^2 - b/x$ where a and b are positive constants and $x>0$. Fully describe the possible motions. For what range(s) of energies is the motion periodic? For what range(s) is it not?

B4. A wheel mounted on an axis that is not frictionless is initially at rest. A constant external torque of 50 N·m is applied to the wheel for 20 s. At the end of the 20 s, the wheel rotates with the frequency 600 rev/min. The external torque is then removed, and the wheel comes to rest after 120 s more.

1. What is the moment of inertia of the wheel?
2. What is the frictional torque, which is assumed to be constant?

Physical Constants

speed of light .... $c = 2.998\times10^{8}$ m/s	electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F
Planck's constant .... $h = 6.626\times10^{-34}$ J·s	electron mass .... $m_{\mathrm{el}} = 9.109\times10^{-31}$ kg
Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s	electron rest energy...... 511.0 keV
Boltzmann constant ...... $k_{\mathrm{B}} = 1.381\times10^{-23}$ J/K	Compton wavelength .. $\lambda_{\mathrm{C}} = h/m_{\mathrm{el}}c = 2.426$ pm
elementary charge .... $e = 1.602\times10^{-19}$ C	proton mass .... $m_{\mathrm{p}} = 1.673\times10^{-27}\,\mathrm{kg} = 1836 m_{\mathrm{el}}$
electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m	1 bohr............................ $a_0 = \hbar^2/ke^2m_{\mathrm{el}} = 0.5292$ Å
magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m	1 hartree (= 2 rydberg) ... $E_{\mathrm{h}} = \hbar^2/m_{\mathrm{el}}a_0^{\,2} = 27.21$ eV
molar gas constant.......... $R = 8.314$ J/(mol·K)	gravitational constant ... $G = 6.674\times10^{-11}$ m^3/(kg·s^2)
Avogadro constant .... $N_{\mathrm{A}} = 6.022\times10^{23}$ mol^{-1}	hc .... $hc = 1240$ eV·nm
Stefan-Boltzmann const.. $\sigma = 5.670\times10^{-8}$ W/(m^2·K^4)	

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

THERMODYNAMICS

Heat capacity = $C_V = N\dfrac{d\langle E\rangle}{dT}$

Clausius' theorem: $\displaystyle\sum_{i=1}^{N}\frac{Q_i}{T_i} \le 0$, which becomes $\displaystyle\sum_{i=1}^{N}\frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Molar heat capacity of diatomic gas is $C_V = \tfrac{5}{2}R$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$dU = TdS - pdV \qquad dF = -SdT - pdV$

$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V, \qquad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p, \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V, \qquad -\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p$$

TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$

(b) Slender rod, axis through one end

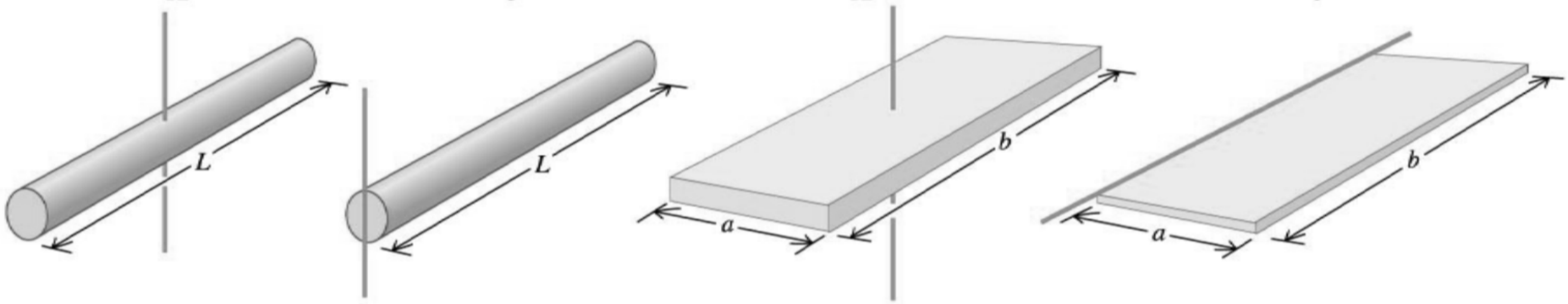
$$I = \frac{1}{3}ML^2$$

(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$

(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$

(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$

(g) Thin-walled hollow cylinder

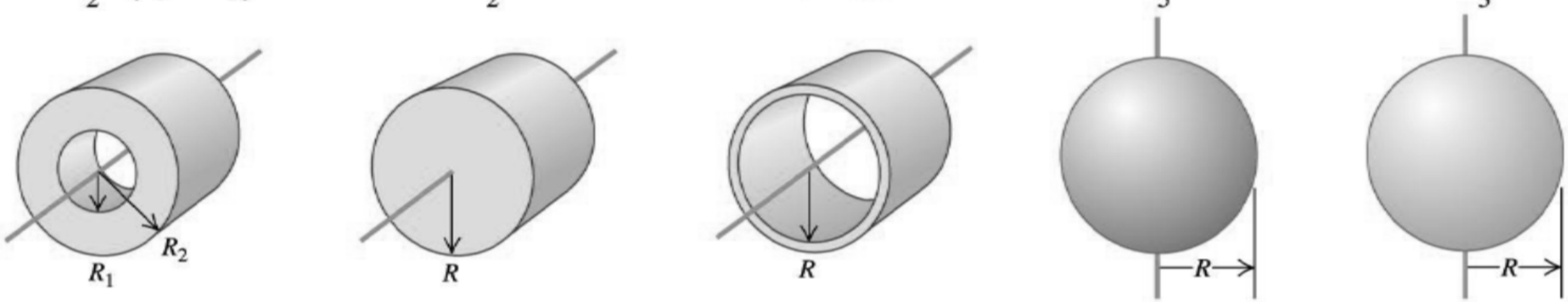
$$I = MR^2$$

(h) Solid sphere

$$I = \frac{2}{5}MR^2$$

(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi/2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$