UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, May 25, 2023

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1. A 600 g copper ball has a temperature of 700°C when it is placed in 3.00 kg of water at a temperature of 20°C. Calculate the temperature (in °C) of the system when equilibrium has been reached. Assume the system is thermally insulated. Data:
$C_{\text{water}} = 4.18\ \text{J g}^{-1}\ \text{K}^{-1}$, $C_{\text{Cu}} = 0.39\ \text{J g}^{-1}\ \text{K}^{-1}$

A2. One mole of diatomic ideal gas ($C_V = 2.5\, nR$) performs a transformation from an initial state for which temperature and volume are, $290\ K$ and $30000\ ml$ to a final state in which temperature and volume are 310 K and $16000\ ml$. The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

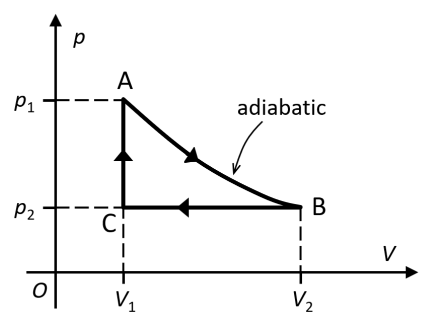
A3. Consider a gas with the following speed probability distribution $f(v) = A$ when $0 < v < v_0, = 0$ otherwise. Find (a) average speed, (b) rms speed in the 1 dimensional, 2 dimensional, and 3 dimensional cases.

A4. An ideal gas (γ= 1.4) expand in an adiabatic process to 10 times of its original volume. If the initial temperature is 0 °C, and the initial pressure 1 atm, find the final temperature and the final pressure.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1. An ideal gas is expanded adiabatically from (p_1,V_1) to (p_2,V_2). It is then compressed isobarically to (p_2,V_1). Finally, the pressure is increased to p_1 at constant volume V_1. Show that the efficiency of the cycle is

$$\eta = 1 - \gamma \frac{V_2 / V_1 - 1}{p_1 / p_2 - 1}$$



B2. The entropy of an ideal gas is $S = n/2\,[a + 5R\, ln(U/n) + 2R\, ln(V/n)]$, where n is the mole number, R is the universal gas constant, U is internal energy, V is volume, and a is a constant.

(a) Calculate the constant pressure heat capacity (C_P) and the constant volume heat capacity (C_V).
(b) Rewrite entropy in (T,V), (T,P), and (P,V) representation.

B3. Consider the adiabatic free expansion of an ideal gas (from volume V to 2V).

(a) What's the work and heat in the process?

(b) How does the temperature change?

(c) Show that this process is irreversible. (Hint: calculate the entropy change)

(d) How would your answer in part (b) change if the gas is not ideal, for example, has a high density?

B4. In an isothermal expansion, an ideal gas at an initial pressure P_0 expands until its volume is twice the initial volume.

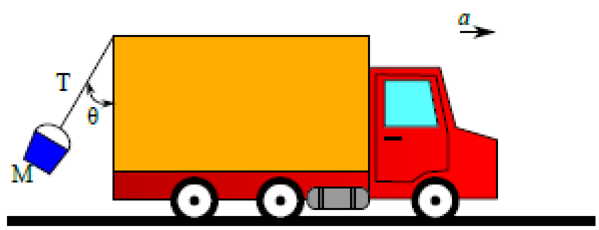
(a) Find its pressure after the expansion.

The gas is then compressed adiabatically and quasi-statically back to its original volume, at which point its pressure is $1.32P_0$.

(b) Is the gas monoatomic, diatomic, or polyatomic?

(c) How does the translational kinetic energy of the gas molecules change in these processes?

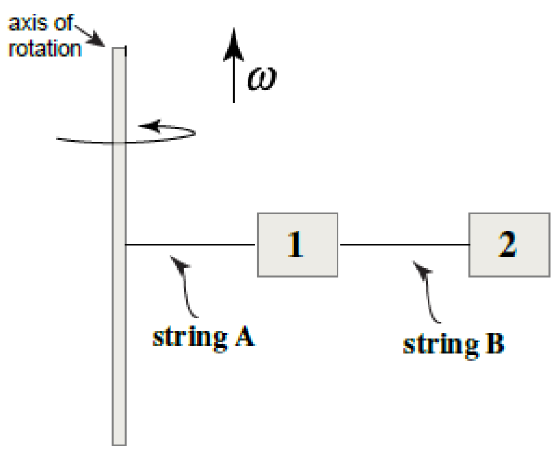
Classical Mechanics Group A - Answer only two Group A questions



A1. A truck is traveling in a straight line on level ground, and is accelerating uniformly with an acceleration of magnitude *a*. A rope (massless and inextensible) is tied to the back of the truck. The other end of the rope is tied to a bucket of mass M. The bucket swings wildly when the truck starts to accelerate, but later on it stays at a fixed position at a fixed distance behind the truck, with the rope hanging straight at a fixed angle, as shown in the figure.

1. Find the angle θ at which the rope will settle. Express your answers in terms of the given variables M, g, and *a* as needed.

2. What will the tension T of the rope be once it settles into this angle? Express your answers in terms of the given variables M, g, and *a* as needed.



A2. Box 1 and box 2 are spinning around a shaft with a constant angular velocity of magnitude ω. Box 1 is at a distance d from the central axis, and box 2 is at a distance 2d from the axis. You may ignore the mass of the strings and neglect the effect of gravity.

(a) Calculate T_B, the tension in string B (the string connecting box 1 and box 2):
(b) Calculate T_A, the tension in string A (the string connecting box 1 and the shaft):
Express your answers in terms of d, ω, m_1 and m_2, the masses of boxes 1 and 2.

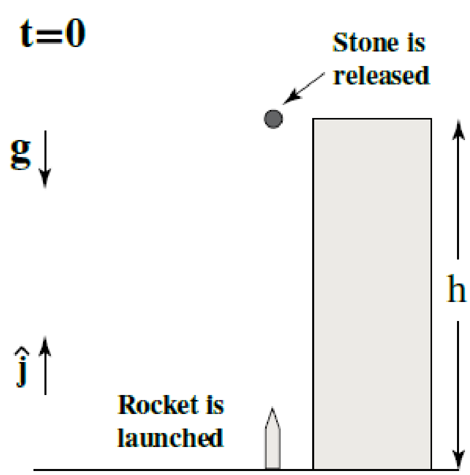
A3. A particle of mass m moves under the influence of a force

$$F(x,t) = \frac{k}{x^2} e^{-t/\tau}$$

where k and τ are positive constants. Find the Lagrangian and Hamiltonian. Is the energy of the system conserved?

A4. On a distant planet, to determine its acceleration due to gravity an astronaut throws a rock straight up with a velocity of +15 m/s and measure a time of 20.0 s before the rock returns to his hand. He then sets up a rotating horizontal disk on which he can sit at any radius. But as the disk begins to speed up, the astronaut with a mass of 80 kg may slide off if the frictional force is insufficient. If the coefficients of static and kinetic friction are respectively 0.4 and 0.3, and the angular velocity is 2 rad/s, what is the maximum radius R where he can sit and still remain on the disk? What is the maximum radius if the astronaut is lighter or heavier by 10 kg?

<u>**Classical Mechanics Group B** - Answer only two Group B questions</u>

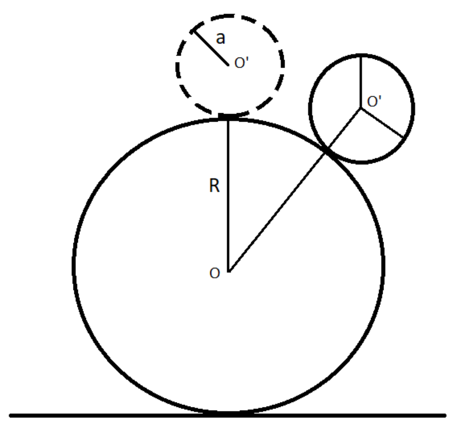


B1. At the base of a vertical cliff, a model rocket, starting from rest, is launched upwards at t = 0 with a time-varying acceleration given by

$$a_y(t) = A - Bt,$$

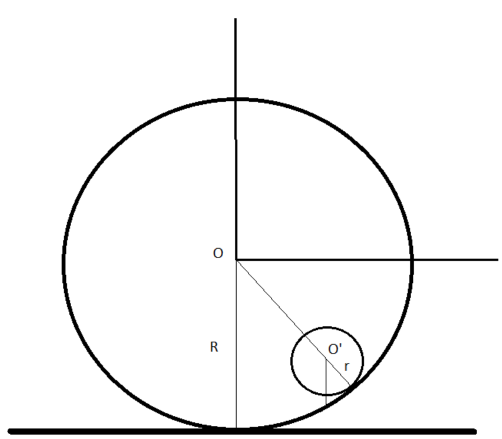
where A and B are positive constants. Also at t = 0, a small stone is released from rest from the top of the cliff at a height h directly above the rocket. (This height h is higher than the maximum height reached by the rocket.) The stone hits the rocket at the instant when the rocket reaches its maximum height. The gravitational acceleration of magnitude g is downward. You may neglect air resistance. Determine an expression for the initial height h from which the stone was dropped in terms of the constants A, B, and g.

B2. If a projectile moves such that its distance from the point of projection is always increasing, find the maximum angle above the horizontal with which the particle could have been projected. Ignore air resistance.



B3. A uniform sphere of radius a is balanced on the top of a perfectly rough fixed cylinder of radius R (a<R). After the balance is slightly disturbed, the sphere rolls down and leaves the cylinder when the line connecting the centers of the cylinder and the sphere makes an angle θ with the vertical.

a) Find θ (you should get a numerical value).
b) Find the Lagrangian of the rolling sphere before it leaves the cylinder.



B4. A solid homogeneous cylinder of radius r and mass m rolls without slipping on the inside of a stationary larger cylinder of radius R. The small cylinder starts at rest from an angle θ_0 from the vertical.

a) Find the velocity of the center of mass of the rolling cylinder.
b) Find the Lagrangian of the rolling cylinder.
c) Find the equation of motion for the rolling cylinder for the angular coordinate θ of its center of mass.
d) For small oscillations about the equilibrium position $\theta = 0$, show that the equation of motion reduces to that for simple harmonic motion and determine its period in terms of R, r, and g.

Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s   Atmospheric pressure.... 101,325 Pa

electron mass .... $m_{\text{el}} = 9.109 \times 10^{-31}$ kg   Avogadro constant .... $N_{\text{A}} = 6.022 \times 10^{23}$ mol^{-1}

Boltzmann constant $k_{\text{B}} = 1.381 \times 10^{-23}$ J/K =8.617x10^{-5} eV/K   gas constant ... $R = 8.314$ J/(mol·K)

Atomic mass unit 1 u=1.66x10^{-27} kg

gravitational constant ..... $G = 6.674 \times 10^{-11}$ m^3/(kg·s^2) g=9.8 m/s^2

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:
$\sin x \approx x - \frac{1}{6}x^3$
$\cos x \approx 1 - \frac{1}{2}x^2$
$\tan x \approx x + \frac{1}{3}x^3$

THERMODYNAMICS

Specific heat of water: 4.186 kJ/(kg K)
Latent heat of fusion for water 333.5 kJ/kg

Heat capacity = $C_V = N\frac{d\langle E\rangle}{dT}$

Clausius' theorem: $\sum_{i=1}^{N}\frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N}\frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Molar heat capacity of diatomic gas is $C_V = \frac{5}{2}R$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dF = -SdT - pdV$$
$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$
$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$ Carnot efficiency = $1 - T_c/T_h$

The Maxwell-Boltzmann distribution function

$$f(v)dv = \frac{4}{\pi^{1/2}}\left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-mv^2/kT} dv$$

TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center

$I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end

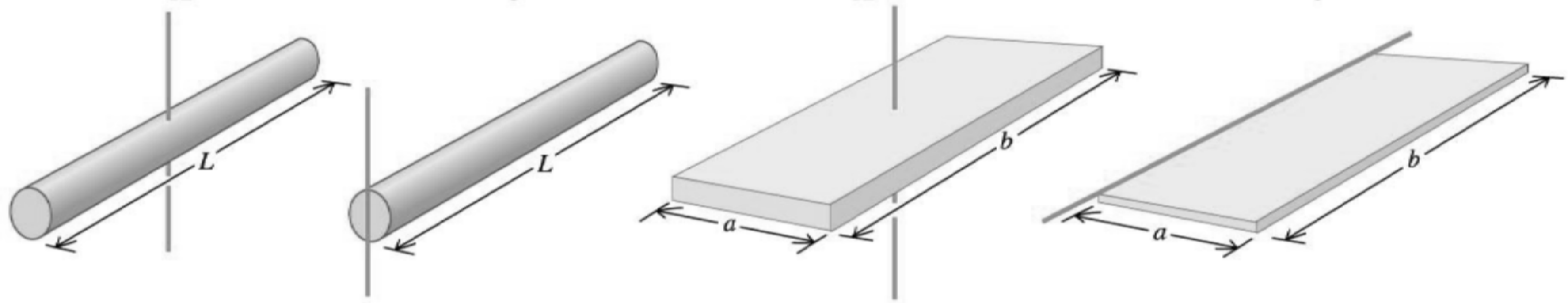
$I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center

$I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge

$I = \frac{1}{3}Ma^2$



(e) Hollow cylinder

$I = \frac{1}{2}M(R_1^2 + R_2^2)$

(f) Solid cylinder

$I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder

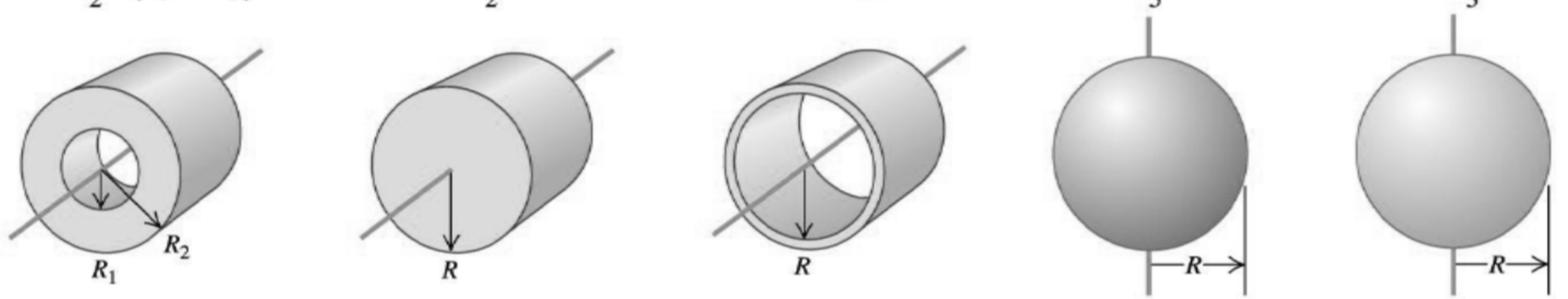
$I = MR^2$

(h) Solid sphere

$I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere

$I = \frac{2}{3}MR^2$



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\frac{1}{2a}$
$x^2e^{-ax^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}\,dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}\,dx = \frac{\frac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$