UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Friday, August 11, 2023

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1. Gamma rays of energy 1.00 MeV are scattered from electrons at rest. The scattering is symmetric, that is the photon scattering angle θ equals the electron scattering angle φ. Find the scattering angle θ and the energy of the scattered photons.

A2. We consider the operators $T_1 = e^A$ and $T_2 = e^{iA}$, where A is hermitian.

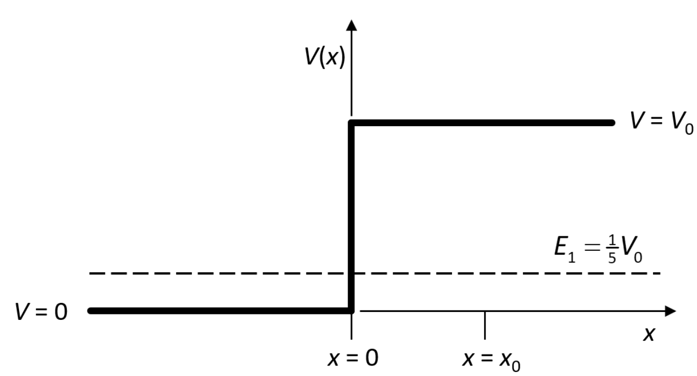
a. Is T_1 hermitian?
b. Is T_2 hermitian?

A3. Derive the Ehrenfest theorem,

$$\frac{d}{dt}\langle\varphi(t)\,|\,L\,|\,\varphi(t)\rangle = \frac{i}{\hbar}\langle\varphi(t)\,|\,HL - LH\,|\,\varphi(t)\rangle,$$

where H is the Hamiltonian, L is an operator (observable) which does not depend on time explicitly, and $\varphi(t)$ is the time-dependent wavefunction.

A4. A particle moves in the potential $V(x)$ shown in the figure. For $x < 0$, the potential is 0. For $x > 0$, it is V_0. The total energy of the particle is $E_1 = \frac{1}{5}V_0$ (dashed line in the figure). Coming from the left, the particle's wavefunction at some position $x = x_0$ (see figure) is $\psi(x = x_0) = \frac{1}{10}\psi(0)$. The total energy is now increased to the value E_2 such that $\psi(x = x_0) = \frac{1}{5}\psi(0)$. Calculate E_2 / V_0.



Quantum Mechanics Group B - Answer only two Group B questions

B1. The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$$

where *a* is a number with the dimension of energy. Find the energy eigenvalues and the corresponding (unnormalized) energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

B2. A particle with mass *m* moves in a delta-function potential

$$V(x) = -V_0 a\delta(x)$$

and has total energy $-E < 0$. Find the particle's stationary wavefunction and the energy *E*.

B3. A system consists of two linearly independent states $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The Hamiltonian has the form $H = \begin{pmatrix} h & k \\ k & h \end{pmatrix}$ where *h* and *k* are real constants. If the system is initially prepared in state $|2\rangle$ at time *t* = 0, what is its state at a later time *t*?

B4. Consider a particle in a state given by the wavefunction $\Psi(x,t) = A\exp(-x^2/a^2)\exp(-iEt/\hbar)$, where *A* is a real number.

a. Find $\langle x\rangle$ and $\langle x^2\rangle$.

b. Assuming this is a state of minimum uncertainty, find the standard deviation of the momentum distribution, σ_p.

<u>Electrodynamics Group A</u> <u>- Answer only two Group A questions</u>

A1.

a. Find the volume charge density $\rho(r)$ creating the electric field $\mathbf{E} = a\mathbf{r}e^{-br}$ where a and b are positive constants.
b. Plot (sketch) $\rho(r)$ over the whole range of r. Find the zero and minimum of $\rho(r)$ and indicate it on the graph.
c. What is the total charge of the system? Answer this question without integrating the charge density found in part (a). Interpret this result in terms of the plot drawn in part (b).

A2. Three point charges, q, $-2q/3$, and $-q/3$, are placed at the vertices of an equilateral triangle with side a. Find the magnitude and direction of the dipole moment **p** of the system in terms of q and a. Indicate the direction by finding the angle between vector **p** and the line joining the second and the third charges.

A3. A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and **r** is the vector from the center.

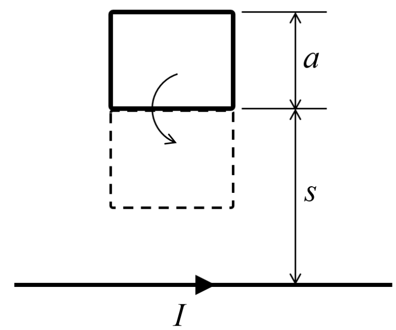
a. Calculate the surface and volume bound charges
b. Calculate the field inside and outside the sphere

A4. A long cylindrical cable carries current in one direction with the current density $J = as$, where a is a constant, and s is the distance from the cylinder's axis. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find

a. The magnetic field as a function of s.
b. The self-inductance per unit length.

Electrodynamics Group B - Answer only two Group B questions

B1. A square loop, side a, resistance R, lies at a distance s ($s > a$) from an infinite straight wire that carries current I. The loop performs a half of revolution about its bottom side arriving into the position indicated by the dashed square.



a. Indicate the direction of the induced current at both positions.

b. What total charge passes a given point in the loop? The answer to part (*a*) is important for this calculation.

B2. The intensity of the radiation from a small light source is 10 W/m^2 at the distance r = 1 m from the source. It may help to remind you that the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$ is the energy transported by the EM fields per unit time per unit area.
Find

a. The total time-averaged radiated power.

b. The electric and magnetic field amplitudes at r = 1 m and r = 3 m.

c. The average energy density at r = 1 m and r = 3 m.

B3. A uniform line charge with the linear charge density λ is placed on an infinite straight wire, a distance d from a grounded conducting plane. Choose a Cartesian coordinate system where the xy plane is the conducting plane, and the wire is parallel to the x axis.

a. Find the potential in the region above the plane at an arbitrary point (x,y,z).

b. Find the surface charge density induced on the conducting plane.

c. Find the total charge on a strip on the conducting plane whose length is infinite in the y direction and whose width is ℓ in the x direction.

B4. The density of electrons in copper is 0.847×10^{23} electrons / cc.

a. Calculate the average electron velocity in a copper wire 1 mm in diameter, carrying a current 1 A.

b. What is the force of attraction per unit length between two such wires, 1 cm apart?

c. Suppose the positive ions are somehow removed from the wire. What is the ratio of the electrical force to the magnetic force you found in (*b*)? Find both an algebraic and a quantitative expression. What happens to this ratio as the current in the wire is increased?

Physical constants

speed of light .... $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant .. $k_B = 1.381\times10^{-23}$ J/K

elementary charge ..... $e = 1.602\times10^{-19}$ C

electric permittivity ... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant...... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022\times10^{23}$ mol^{-1}

fine structure constant $\alpha = ke^2/(\hbar c)$

electrostatic const.... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times10^{-31}$ kg

electron rest energy 511.0 keV

Compton wavelength $\lambda_C = h/m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673\times10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 Ry) $E_h = \hbar^2/m_{el}a_0{}^2 = 27.21$ eV

gravitational constant $G = 6.674\times10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

1 Ry =13.6 eV

Equations That May Be Helpful

TRIGONOMETRY

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$\cos(ix) = \cosh(x)$

$\sin(ix) = i\sinh(x)$

For small x:

$\sin x \approx x - \tfrac{1}{6}x^3$

$\cos x \approx 1 - \tfrac{1}{2}x^2$

$\tan x \approx x + \tfrac{1}{3}x^3$

QUANTUM MECHANICS

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x, L_y] = i\hbar L_z \quad et\ cycl.$

Ladder operators:
$$L_+ \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\, \ell, m+1\rangle$$
$$L_- \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\, \ell, m-1\rangle$$

Gyromagneic ratio for electron (SI units) = e/m

Compton formula

$$\lambda' - \lambda = \lambda_C(1-\cos\theta)$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z)=\frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z)=\sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z)=\mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z)=\sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z)=\mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z)=\mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

Stationary states of harmonic oscillator for n=0 and n=1

$$\varphi_0(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\alpha^2 x^2/2}$$

$$\varphi_1(x) = \left(\frac{\alpha}{2\pi^{1/2}}\right)^{1/2} 2\alpha x\, e^{-\alpha^2 x^2/2}$$

where $\alpha = (m\omega/\hbar)^{1/2}$

Ladder operators for harmonic oscillator

$$a_\pm = \frac{1}{\sqrt{2}}\left(\alpha x \mp i\frac{p}{\hbar\alpha}\right)$$

Radial functions for the hydrogen atom $R_{nl}(r)$

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\exp(-r/a_0) \qquad R_{20}(r) = \frac{2}{(2a_0)^{3/2}}[1 - r/(2a_0)]\exp[-r/(2a_0)]$$

$$R_{21}(r) = \frac{r}{24^{1/2}a_0^{5/2}}\exp[-r/(2a_0)]$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q[V(\mathbf{b}) - V(\mathbf{a})]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau = Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$, in which

$q = \int \rho(\mathbf{r})d^3\mathbf{r}$ is the monopole moment

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}d^3\mathbf{r}$ is the dipole moment

$Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right]d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$)

Relative permittivity: $\varepsilon_r = 1 + \chi_e$

Bound charges

$\rho_{\mathrm{b}} = -\nabla \cdot \mathbf{P}$

$\sigma_{\mathrm{b}} = \mathbf{P} \cdot \hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0 \dfrac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

Relative permeability: $\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}}$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$ Current densities: $I = \int \mathbf{J} \cdot d\mathbf{A}$, $I = \int \mathbf{K} \cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \int \dfrac{I d\boldsymbol{\ell} \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\mathrm{encl}}$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I \int d\mathbf{a}$

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \dfrac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$

Bound currents

$J_{\mathrm{b}} = \nabla \times \mathbf{M}$

$K_{\mathrm{b}} = \mathbf{M} \times \hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \dfrac{\partial \mathbf{D}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1}\int_V B^2 d\tau = \frac{1}{2}LI^2 = \frac{1}{2}\oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

Wave equations in a conducting medium

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}.$$

Boundary conditions in electrodynamics

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$$

$$\int_0^\infty x^n e^{-x} dx = n!$$

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2 e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4 e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6 e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2} dx = \ln\left(x + \sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1} dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2} dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2} dx = \frac{\dfrac{bx}{x^2+b^2} + \arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$