UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 11, 2022

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1. One mole of nitrogen gas is heated from t=20 C to t=100 C
- (a) How much heat must be supplied if the volume is kept constant?
- (b) How much heat must be supplied if the pressure is kept constant?
- (c) How much work is done by the gas in part (b)?

A2 (a) Consider the reversible isothermal expansion of n moles of an ideal gas in contact with a heat reservoir at temperature T. The volume of the gas doubles during the expansion process. Calculate the entropy change of the gas.
(b) Consider now the free expansion of the same amount of ideal gas taking place inside an insulated container with rigid walls after an internal partitioning wall is punctured. Again the volume of the ideal gas doubles. Find the entropy change of the gas.

A3. Three factories (A, B, and C) manufacture batteries. Factory A produces 20% of the batteries and factory B produces 75% of the batteries. The remaining 5% of the batteries are from factory C. The defective rate for factory A is 1 in 50, the defective rate for factory B is 1 in 20, and the defective rate for factory C is 1 in 100. Given that a randomly chosen battery is defective, what is the probability that it came from factory C?

A4. The size of the oxygen molecule is about $2.0\text{x}10^{-10}$ m. Make a rough estimate of the pressure at which the finite volume of the molecules should cause noticeable deviations from ideal-gas behavior at T=300 K.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1. 500 grams of ice cubes at 0 C are placed in 1 liter of water at 30 C. The system then comes to equilibrium with no heat exchange with the surroundings.
(a) Does the ice melt completely? If yes, find the temperature of the water in equilibrium. If not, find how much ice remains in equilibrium.
(b) Calculate the total change of entropy for the whole system.

B2. The equation of state of some material is

$$pV = AT^3,$$

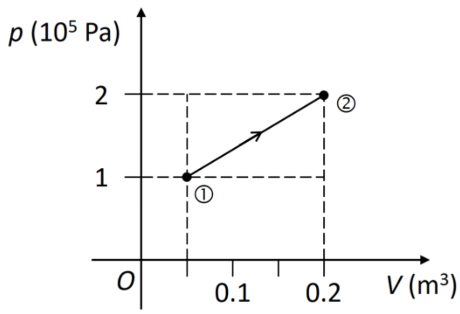
where p, V, and T are the pressure, volume, and temperature, respectively, and A is a constant. The internal energy of the material is

$$U = BT^n \ln\left(\frac{V}{V_0}\right) + f(T),$$

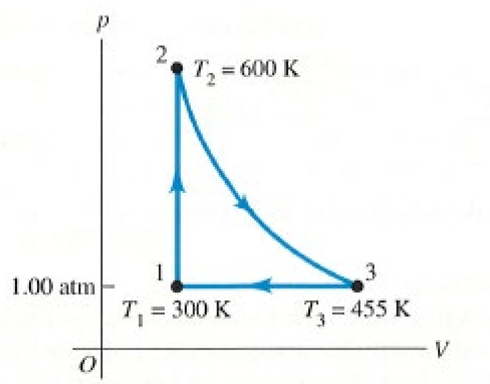
where B, n, and V_0 are all constants, and $f(T)$ only depends on the temperature.

(a) For a given increments dV and dT, find the increment of entropy, dS
(b) Find parameters B and n. Hint: use the fact that dS is a complete differential.

B3. One mole of an ideal diatomic gas is taken from state with $P_1=10^5$ Pa, $V_1=0.05$ m^3 to state with $P_2=2\text{x}10^5$ Pa, $V_2=0.20$ m^3, along a path that is a straight line in the *PV* diagram. Find the change of the gas's internal energy, the work done by the gas, and the heat transferred to the gas.



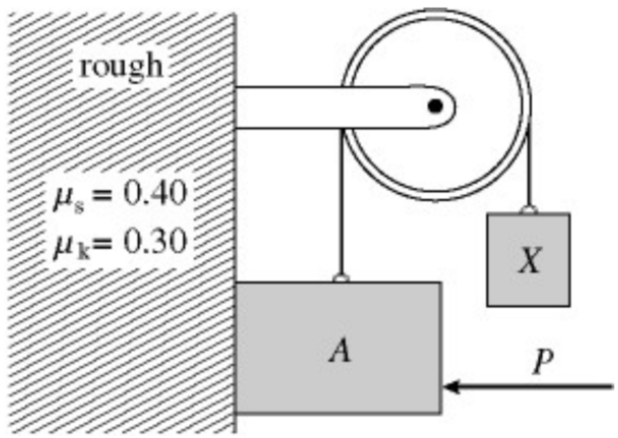
B4. A heat engine takes 0.300 mol of an ideal diatomic gas around the cycle shown in the *pV* diagram. Process 1->2 is isohoric, process 2->3 is adiabatic, and the process 3->1 is isobaric at $P=10^5$ Pa.



(a) Find the pressure and volume at points 1, 2, and 3;
(b) Calculate Q, W and ΔU for each of the three processes;
(c) Find the net work done by the gas in the cycle;
(d) What is the thermal efficiency of the engine? How does this compare to the efficiency of a Carnot-cycle engine operating between the same minimum and maximum temperatures?

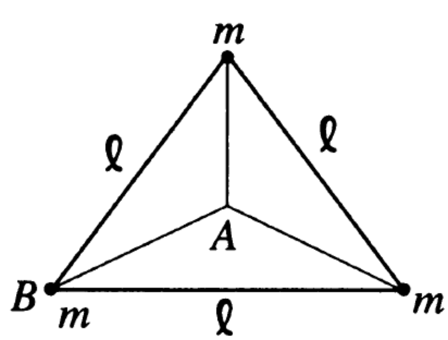
Classical Mechanics Group A - Answer only two Group A questions

A1.



Block A of mass 8 kg and block X are attached to a rope that passes over a pulley. A 50 N force *P* is applied horizontally to block A, keeping it in contact with a rough vertical face. The coefficients of static and kinetic friction between the wall and block A are $\mu_s = 0.4$, $\mu_k = 0.3$. The pulley is light and frictionless. Determine the mass of block X such that block A descends at a constant velocity of 5 cm/s when it is set into motion.

A2.



Problem A2

Three equal masses *m* are rigidly connected to each other by massless rods of length *l* forming an equilateral triangle, as shown in the figure above. The assembly is given an angular velocity ω about an axis perpendicular to the triangle. For fixed ω, determine the ratio of the kinetic energy of the assembly for an axis through *B* compared with that for an axis through *A*.

A3. Two cars start 200 m apart and drive toward each other at 10 m/s. A grasshopper jumps back and forth between the cars with a constant horizontal speed of 15 m/s relative to the ground. The grasshopper jumps the instant he lands, so he spends no time resting on either car. What distance does the grasshopper travel before the cars collide?

A4. A particle of mass 1 kg undergoes one-dimensional motion such that its velocity varies according to $v(x) = \beta x^{-n}$, where β and *n* are constants and *x* is the position of the particle. Determine the acceleration of the particle as a function of its position *x*.

Classical Mechanics Group B - Answer only two Group B questions

B1. A particle of mass *m* undergoes one-dimensional motion in a potential

$$U(r) = U_0\left(\frac{r}{R} + \frac{\lambda^2 R}{r}\right)$$

where *r* is the distance from the origin, $0 \le r \le \infty$. The quantities U_0, *R*, *λ* are positive constants. Find the equilibrium position r_0. For small displacements *x* from this equilibrium point, show that the potential is quadratic in *x*. Find the frequency of small oscillations.

B2. A short uniform solid cylinder has a few turns of light string wound around it. The end of the string is held steady and the cylinder is allowed to fall under gravity with its axis remaining parallel to the surface.

(a) Using Newtonian mechanics, find the acceleration of the center of the cylinder and the tension in the string;

(b) Write down the Lagrangian of the system, and obtain the acceleration by solving the Lagrangian equation.

B3. A particle of mass *m* is moving in the three-dimensional central potential

$$U(r) = \frac{kr^2}{2}$$

(a) For a given a angular momentum *L,* find the lowest possible energy of the particle E_{min}; what is the shape of the particle's orbit for $E = E_{min}$?

(b) For a given particle's energy $E > E_{min}$, find the distance of the closest approach to the center and the farthest distance from the center.

(c) Explain why the particle's orbit is in general planar. Given the fact that any orbit in the given field is closed, consider the motion of the particle in a two-dimensional (say, *xy*) plane and obtain the period of the motion.

B4. A uniform solid sphere of mass *m* and radius *a* is rolling down an inclined plane without slipping. The inclination angle is θ.

(a) Write down the Lagrangian of the system and the Lagrange's equation;

(b) From Lagrange's equation, find the linear acceleration of the sphere;

(c) Write down the Hamiltonian of the system and the Hamilton's equations;

(d) Write down Hamilton's equations and obtain the linear acceleration of the sphere.

Physical Constants

speed of light .... $c = 2.998\times10^{8}$ m/s

Atmospheric pressure.... 101,325 Pa

electron mass .... $m_{\mathrm{el}} = 9.109\times10^{-31}$ kg

Avogadro constant .... $N_{\mathrm{A}} = 6.022\times10^{23}$ mol^{-1}

Boltzmann constant $k_{\mathrm{B}} = 1.381\times10^{-23}$ J/K $=8.617\times10^{-5}$ eV/K

gas constant ... $R = 8.314$ J/(mol·K)

Atomic mass unit 1 u$=1.66\times10^{-27}$ kg

gravitational constant ..... $G = 6.674\times10^{-11}$ m^3/(kg·s^2) g=9.8 m/s^2

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

THERMODYNAMICS

Specific heat of water: 4.186 kJ/(kg K)

Latent heat of fusion for water 333.5 kJ/kg

Heat capacity = $C_V = N\dfrac{d\langle E\rangle}{dT}$

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Molar heat capacity of diatomic gas is $C_V = \frac{5}{2}R$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dF = -SdT - pdV$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$ Carnot efficiency = $1 - T_c/T_h$

The Maxwell-Boltzmann distribution function

$$f(v)dv = \frac{4}{\pi^{1/2}}\left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-mv^2/kT} dv$$

TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$

(b) Slender rod, axis through one end

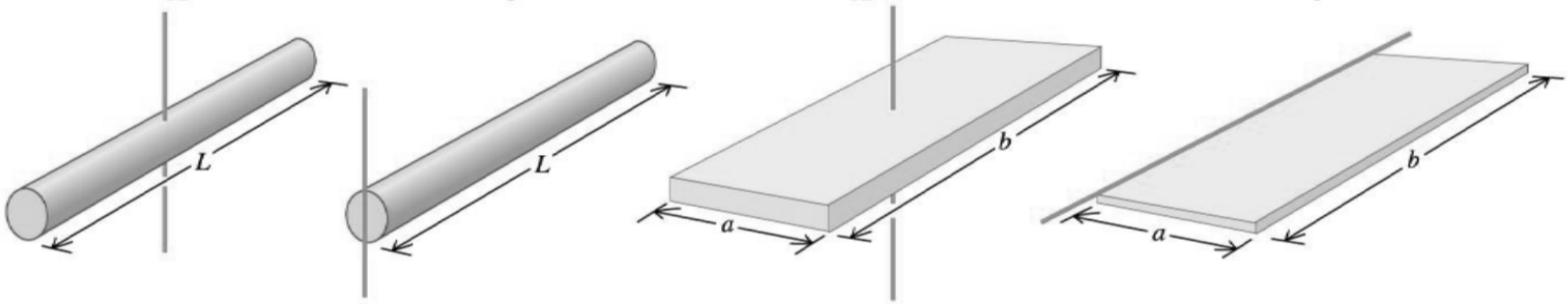
$$I = \frac{1}{3}ML^2$$

(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$

(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$

(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$

(g) Thin-walled hollow cylinder

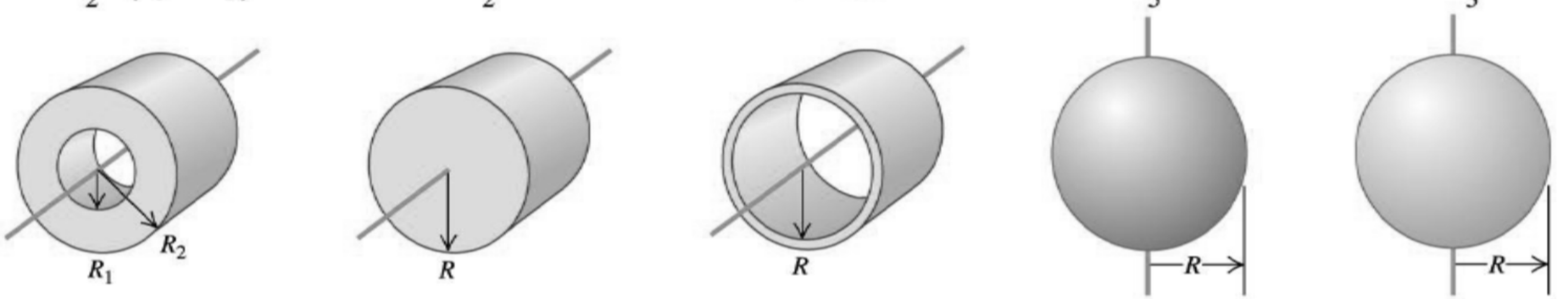
$$I = MR^2$$

(h) Solid sphere

$$I = \frac{2}{5}MR^2$$

(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
$e^{-a x^2}$	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
$x e^{-a x^2}$	$\dfrac{1}{2a}$
$x^2 e^{-a x^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-a x^2}$	$\dfrac{1}{2a^2}$
$x^4 e^{-a x^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-a x^2}$	$\dfrac{1}{a^3}$
$x^6 e^{-a x^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2} dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1} dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2} dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2} dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$