EM Problems

B1. A long coaxial cable carries current I such that it flows uniformly down the surface of the inner cylinder of radius a and goes back along the outer cylinder of radius b, as shown in the figure below. Assuming that the two conductors have the potential difference V and the space between the wires is filled by the dielectric material of dielectric permittivity ε,

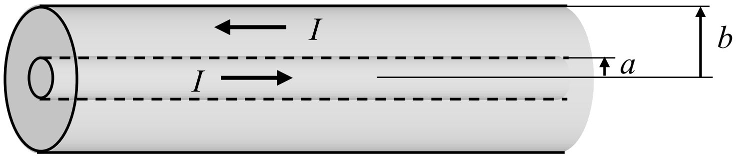
5 1. find the magnitude of the linear change λ on the inner wire;

5 2. find the electric field $\mathbf{E}$ everywhere in space;

5 3. calculate the electric polarization $\mathbf{P}$ related to the electric displacement $\mathbf{D}$ by $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$;

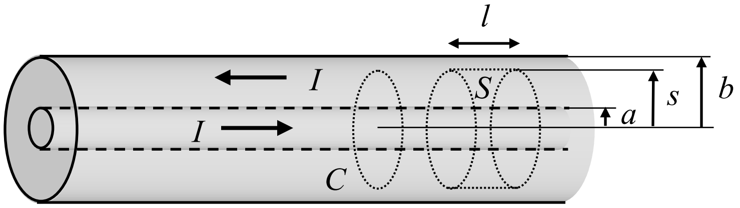
5 4. calculate the bound charge densities (ρ_B and σ_B) induced by the polarization $\mathbf{P}$. Show that the total bound charge is zero;

5 5. find the magnetic field $\mathbf{B}$ everywhere.



Useful formula: $\nabla \cdot \mathbf{P} = \frac{1}{s}\frac{\partial}{\partial s}(sP_s) + \frac{1}{s}\frac{\partial P_\phi}{\partial \phi} + \frac{\partial P_z}{\partial z}$

Solution:



1. The potential difference implies that there is an electric field between the conductors which is associated with free charges on the conductors. Due to symmetry of the problem the charge is distributed uniformly. Assume that the linear change density of the inner conductor is λ. Then the electric field can be found from the Gauss's law which in the presence of dielectric material says that

$$\oint_S \mathbf{D} \cdot \mathbf{n} da = Q,$$

where **D** is the electric displacement, and Q is the total free charge enclosed by surface S. By integrating over the cylinder shown in the figure above and taking into account that due to symmetry the electric displacement **D** is pointed along the radius $\hat{\mathbf{s}}$ we find that

$$\int_S \mathbf{D}\cdot d\mathbf{a} = D2\pi sl = \lambda l\,,$$

which gives

$$\mathbf{D} = \frac{\lambda}{2\pi}\frac{\hat{\mathbf{s}}}{s}.$$

Electric field **E** is therefore

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\lambda}{2\pi\varepsilon}\frac{\hat{\mathbf{s}}}{s}.$$

But we know that the potential difference V is determined by the electric field so that

$$V = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\varepsilon}\int_a^b \frac{ds}{s} = \frac{\lambda}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right).$$

This gives the linear charge density of

$$\lambda = -2\pi\varepsilon V \ln^{-1}\left(\frac{b}{a}\right).$$

2. The electric field inside the cable is therefore

$$\mathbf{E} = -V\ln^{-1}\left(\frac{b}{a}\right)\frac{\hat{\mathbf{s}}}{s}.$$

Outside the cable the electric field is zero because the linear charge on the outer cylinder is opposite to that on this inner cylinder.

3. The electric field **E** produces electric polarization **P** and electric displacement **D** which are related as follows

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}.$$

Therefore, we have

$$\mathbf{P} = (\varepsilon-\varepsilon_0)\mathbf{E} = -(\varepsilon-\varepsilon_0)V\ln^{-1}\left(\frac{b}{a}\right)\frac{\hat{\mathbf{s}}}{s}.$$

4. The bulk bound charge density is given by the polarization divergence

$$\rho_B = -\nabla\cdot\mathbf{P} = \frac{1}{s}\frac{\partial}{\partial s}\left(sP_s\right) = (\varepsilon-\varepsilon_0)V\ln^{-1}\left(\frac{b}{a}\right)\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{1}{s}\right) = 0.$$

The surface bound charge density on the inner cylinder is

$$\sigma_B = \mathbf{P}\cdot\hat{\mathbf{s}}\Big|_{s=a} = -\frac{(\varepsilon-\varepsilon_0)V}{a}\ln^{-1}\left(\frac{b}{a}\right).$$

The surface bound charge density on the outer cylinder is

$$\sigma_B = -\mathbf{P}\cdot\hat{\mathbf{s}}\Big|_{s=b} = \frac{(\varepsilon-\varepsilon_0)V}{b}\ln^{-1}\left(\frac{b}{a}\right).$$

It is easy to see that the total bound change is zero by calculating the linear bound change on the inner and outer cylinders. By integrating the surface bound charges over the circles of radius a and b respectively we find for the inner cylinder

$$\lambda_B = -2\pi(\varepsilon-\varepsilon_0)V\ln^{-1}\left(\frac{b}{a}\right),$$

and for the outer cylinder

$$\lambda_B = 2\pi(\varepsilon-\varepsilon_0)V\ln^{-1}\left(\frac{b}{a}\right),$$

which implies that the total bound charge is zero.

5. The magnetic field can be calculated using the Ampere's law. Applying it to a circle C shown in the figure above and using the symmetry of the problem we find for the region inside the cable

$$\oint_S \mathbf{B}\cdot d\mathbf{l} = 2\pi sB(s) = \mu_0 I\,,$$

which gives

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}\,.$$

Outside the cable the magnetic field is zero due to currents on the outer and inner cylinders being the same magnitude but flowing in the opposite directions.

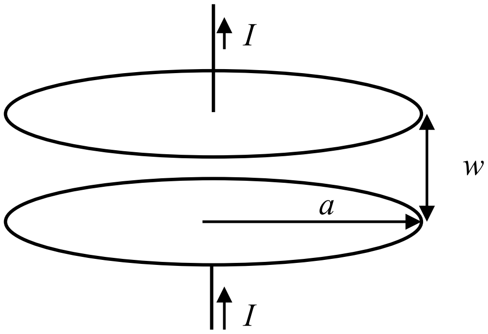
B**2.** A parallel-plate capacitor (see a figure below) is charging such that the electric current I is constant in time and the surface charge σ is uniform at any given time and is zero at $t = 0$.

8 1. Find an electric field **E** between the plates.

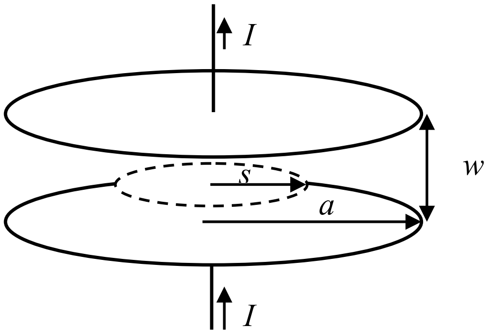
9 2. Find a magnetic field **B** between the plates.

8 3. Find the magnitude and direction of an EM energy flow given by Poynting's vector **S**.

Assume that the separation between the plates w is much less than the radius of the capacitor a ($w << a$) and neglect effects of fringe fields.



Solution:



1. The electric field in the capacitor is given by

$$\mathbf{E}(t) = \frac{\sigma(t)}{\varepsilon_0}\hat{\mathbf{z}},$$

where the time-dependent surface change density is

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}.$$

Therefore,

$$\mathbf{E}(t) = \frac{It}{\pi\varepsilon_0 a^2}\hat{\mathbf{z}}.$$

2. The displacement current through a surface bound by a circle of radius s (shown in the figure by a dashed line) is given by

$$I_d = \varepsilon_0 \frac{dE}{dt} \pi s^2 = I \frac{s^2}{a^2}.$$

The magnetic field is obtained from

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_d ,$$

which leads to

$$B 2\pi s = \mu_0 I \frac{s^2}{a^2}$$

and therefore

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi a^2} s \hat{\phi} .$$

3. The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi \varepsilon_0 a^2} \frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = -\frac{I^2 t}{2\pi^2 \varepsilon_0 a^4} s \hat{\mathbf{s}} ,$$

i.e. the EM energy flows towards the axis of the capacitor.

B3. A metal hollow sphere of radius R is kept under a constant potential Φ_0. Find the electric field $\mathbf{E}$ and the electrostatic potential Φ inside and outside the sphere and determine the surface charge density σ

12 1. by using Gauss's law;

13 2. by solving Laplace's equation.

Show that the solutions are identical.

Useful formula: $\nabla^2\Phi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$.

Solution:

1. Assume that the sphere has surface change σ. According to Gauss's law we have for the electric field outside the sphere

$$\oiint_S \mathbf{E}\cdot d\mathbf{a} = \frac{4\pi R^2\sigma}{\varepsilon_0}.$$

Using for the surface S the sphere of radius r>R and using symmetry of the problem we find

$$\mathbf{E} = \frac{\sigma R^2}{\varepsilon_0 r^2}\hat{\mathbf{r}}.$$

Hence the electrostatic potential outside the sphere is

$$\Phi = -\int_{-\infty}^{r}\mathbf{E}\cdot d\mathbf{r} = -\frac{\sigma R^2}{\varepsilon_0}\int_{-\infty}^{r}\frac{1}{r^2}dr = \frac{\sigma R^2}{\varepsilon_0 r}.$$

The surface change density can be found from the given potential on the sphere which leads to

$$\sigma = \frac{\varepsilon_0\Phi_0}{R}.$$

Therefore outside the sphere the electric field is

$$\mathbf{E}(r) = \frac{\Phi_0 R}{r^2}\hat{\mathbf{r}},$$

and the potential is

$$\Phi = \frac{\Phi_0 R}{r}.$$

Inside the sphere r<R Gauss's law theorem says that the electric field is zero and consequently the electrostatic potential is constant

$$\Phi(r) = \Phi_0.$$

2. Laplace's equation away from the surface says that

$$\nabla^2\Phi = 0.$$

Using the spherical symmetry of the problem we can write

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\Phi(r)\right] = 0.$$

A general solution of this equation is

$$\Phi(r) = \frac{A}{r} + B,$$

where A and B are some constants which are to be found according to boundary conditions. Outside the sphere, r>R, since $\Phi \to 0$ at infinity, we find that B = 0. At r=R the potential $\Phi(R) = \Phi_0$, so that $A = \Phi_0 R$. This leads to

$$\Phi(r) = \frac{\Phi_0 R}{r}.$$

The electric field is given by

$$\mathbf{E}(r) = -\nabla\Phi(r) = \frac{\partial}{\partial r}\left(\frac{\Phi_0 R}{r}\right)\hat{\mathbf{r}} = \frac{\Phi_0 R}{r^2}\hat{\mathbf{r}}.$$

Inside the sphere A=0 (otherwise the potential would diverge at r=0 which is unphysical) and B=Φ_0 due to the boundary condition of the surface. This gives the potential inside the sphere

$$\Phi(r) = \Phi_0$$

and consequently the electric field $\mathbf{E} = 0$. The surface charge can be found from discontinuity of the normal component of the electric field on the boundary. This gives

$$\sigma = \frac{\varepsilon_0 \Phi_0}{R}.$$

Apparently that the results obtained by the two methods are identical.

B4. Consider electromagnetic waves in for free space in the form:

$$\mathbf{E}(x,y,z,t)=\mathbf{E}_0(x,y)e^{i(kz-\omega t)}$$
$$\mathbf{B}(x,y,z,t)=\mathbf{B}_0(x,y)e^{i(kz-\omega t)}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ lie in the *xy*-plane.

15 1. Using Maxwell's equations, find the relation between *k* and *ω*.

5 2. Show that $\mathbf{E}_0$, $\mathbf{B}_0$, and $\hat{\mathbf{z}}$ are mutually orthogonal, and find the relation between $|\mathbf{E}_0|$ and $|\mathbf{B}_0|$.

5 3. Show that $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$ satisfy the equations of electrostatic and magnetostatic in free space.

Solution:

1.
$$\nabla\times\mathbf{E}=-\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=-\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & ik\\ E_{0x} & E_{0y} & 0\end{vmatrix}e^{i(kz-\omega t)}=$$
$$=\left[-ikE_{0y}\hat{\mathbf{x}}+ikE_{0x}\hat{\mathbf{y}}+\left(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}\right)\hat{\mathbf{z}}\right]e^{i(kz-\omega t)}=\left(ik\hat{\mathbf{z}}\times\mathbf{E}_0+\nabla\times\mathbf{E}_0\right)e^{i(kz-\omega t)}.$$

Similarly, $\nabla\times\mathbf{B}=\left(ik\hat{\mathbf{z}}\times\mathbf{B}_0+\nabla\times\mathbf{B}_0\right)e^{i(kz-\omega t)}$.

Maxwell's equations, $\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$, $\nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$, have therefore form:

$$ik\hat{\mathbf{z}}\times\mathbf{E}_0=i\omega\mathbf{B}_0-\nabla\times\mathbf{E}_0,\quad ik\hat{\mathbf{z}}\times\mathbf{B}_0=-\frac{i\omega}{c^2}\mathbf{E}_0-\nabla\times\mathbf{B}_0.$$

Noting that $\nabla\times\mathbf{E}_0$ and $\nabla\times\mathbf{B}_0$ have only *z* components, while $\nabla\times\mathbf{E}_0$ and $\hat{\mathbf{z}}\times\mathbf{B}_0$ are in the *xy* plane we obtain: $\nabla\times\mathbf{E}_0=0$, $\nabla\times\mathbf{B}_0=0$, and

$$\hat{\mathbf{z}}\times\mathbf{E}_0=\frac{\omega}{k}\mathbf{B}_0\ (*),\ \hat{\mathbf{z}}\times\mathbf{B}_0=-\frac{\omega}{kc^2}\mathbf{E}_0\ (**).$$

Taking the vector product of $\hat{\mathbf{z}}$ and (*), we have $\mathbf{E}_0=-\frac{\omega}{k}\hat{\mathbf{z}}\times\mathbf{B}_0$. Substitution into (**) gives:

$$\frac{\omega^2}{k^2c^2}=1\ \text{or}\ k=\frac{\omega}{c}.$$

2. Equations (*) and (**) relate $\mathbf{E}_0$ and $\mathbf{B}_0$, showing that $\mathbf{E}_0$, $\mathbf{B}_0$, and $\hat{\mathbf{z}}$ are mutually perpendicular forming a right-hand set. Their amplitudes are related: $|\mathbf{E}_0|=c|\mathbf{B}_0|$.

3. Maxwell's equations, $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, give: $\nabla \cdot \mathbf{E}_0 = 0$ and $\nabla \cdot \mathbf{B}_0 = 0$.
These equations, together with $\nabla \times \mathbf{E}_0 = 0$ and $\nabla \times \mathbf{B}_0 = 0$, imply that $\mathbf{E}_0(x, y)$ and $\mathbf{B}_0(x, y)$ satisfy the equations of electrostatic and magnetostatic in free space.

A1. Two parallel electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ are separated by distance a and oriented perpendicular to the line connecting them.

15 (1) Find the magnitude and direction of the electric field which is produced by dipole $\mathbf{p}_1$ at a position of $\mathbf{p}_2$.

7 (2) Find the electrostatic energy corresponding to the given dipole configuration.

3 (3) Calculate the work which must be done to reverse the orientation of one of the dipole moments.

Hint: Use the expression for the electrostatic potential which is produced by a dipole $\mathbf{p}$ at point $\mathbf{r}$: $\Phi(\mathbf{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{p}\cdot\mathbf{r}}{r^3}$.

Solution:

(1) The electric field produced by a dipole is

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^3}\nabla(\mathbf{p}\cdot\mathbf{r}) + (\mathbf{p}\cdot\mathbf{r})\nabla\left(\frac{1}{r^3}\right)\right].$$

In Cartesian coordinates the i component of $\nabla(\mathbf{p}\cdot\mathbf{r})$ is

$$\nabla_i(\mathbf{p}\cdot\mathbf{r}) = \frac{\partial}{\partial x_i}\sum_j p_j x_j = \sum_j p_j \frac{\partial x_j}{\partial x_i} = \sum_j p_j \delta_{ij} = p_i,$$

which implies that

$$\nabla(\mathbf{p}\cdot\mathbf{r}) = \mathbf{p}.$$

Taking into account the fact that $\nabla\left(\dfrac{1}{r^3}\right) = -\dfrac{3\mathbf{r}}{r^5}$, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r}) - r^2\mathbf{p}}{r^5}\right].$$ (IF: give them this eq. in the cheat sheet to make it easier, then the derivation is unnecessary).

The electric field which is produced by dipole $\mathbf{p}_1$ at a position of dipole $\mathbf{p}_2$ is therefore:

$$\mathbf{E}_1(\mathbf{a}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\mathbf{a}(\mathbf{p}_1\cdot\mathbf{a}) - a^2\mathbf{p}_1}{a^5}\right] = -\frac{\mathbf{p}_1}{4\pi\varepsilon_0 a^3},$$

where $\mathbf{a}$ is the radius vector connecting the dipoles and we took into account that $\mathbf{p}_1\cdot\mathbf{a} = 0$.

(2) The electrostatic energy of a dipole $\mathbf{p}$ in an external electric field $\mathbf{E}$ is

$$E_{el} = -\mathbf{p}\cdot\mathbf{E}.$$

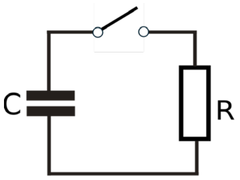
In our case therefore

$$E_{el} = \mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r}) = \frac{\mathbf{p}_2 \cdot \mathbf{p}_1}{4\pi\varepsilon_0 a^3} = \frac{p_1 p_2}{4\pi\varepsilon_0 a^3} .$$

(3) The work which must be done to reverse the orientation of one of the dipole moments is equal to the difference in the electrostatic energy corresponding to the two opposite dipole configurations, i.e.

$$W = \frac{p_1 p_2}{2\pi\varepsilon_0 a^3} .$$

A2. An RC circuit consists of a resistor with resistance *R* and a charged capacitor with capacitance *C* and voltage V_0, across the plates. At time t = 0 the switch is closed.



12 (1) Calculate the electric current flowing in the circuit as a function of time.

13 (2) Show that the initial electrical energy stored in the capacitor is fully dissipated in the resistor at t = ∞.

Solution:

(1) When the switch is closed, the capacitor will discharge its stored energy through the resistor. If *V*(*t*) is to be the voltage of the capacitor's top plate relative to its bottom plate, then the current *I*(*t*) exiting the capacitor's top plate will equal to $I(t) = -\frac{dQ(t)}{dt} = -C\frac{dV(t)}{dt}$. Kirchhoff's current law says this current is the same current entering the top side of the resistor, which per Ohm's law equals $I(t) = RV(t)$. This yields a linear differential equation:

$$-C\frac{dV(t)}{dt} = RV(t).$$

Considering $V(0) = V_0$ the solution of this equation yields:

$$V(t) = V_0 e^{-\frac{t}{RC}},$$

and therefore

$$I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}.$$

(2) The initial energy stored in the capacitor is $W_C = \frac{CV_0^2}{2}$.

The energy dissipated in the resistor is

$$W_R = \int_0^\infty I^2(t)R dt = \int_0^\infty \frac{V_0^2}{R} e^{-\frac{2t}{RC}} dt = -\frac{V_0^2}{R}\frac{RC}{2} e^{-\frac{2t}{RC}}\Bigg|_0^\infty = \frac{CV_0^2}{2}.$$

We therefore have $W_C = W_R$, which implies that the initial electrical energy stored in the capacitor is fully dissipated in the resistor.

EM

A3

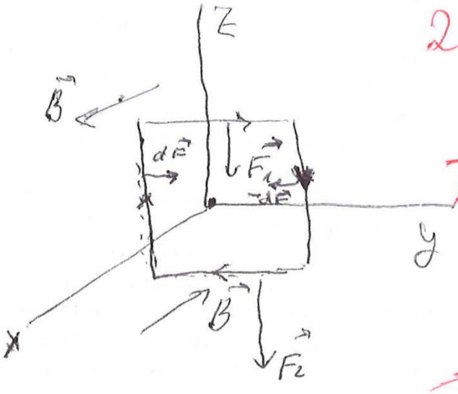
2

From

$$\vec\nabla \times \vec B = \mu_0 \vec J$$

4 (a) $$\vec J = \frac{1}{\mu_0}\vec\nabla \times (kz\hat x) = \frac{k}{\mu_0}\begin{vmatrix} \hat x & \hat y & \hat z \\ \cancel{kz}\,\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & 0 \end{vmatrix} = \frac{k}{\mu_0}\hat y$$

(b)



2 $$\vec F = I\,\vec l \times \vec B$$

forces on sides ∥ to z axis

cancel each other

7 Show $d\vec F$ and $-d\vec F$

$$\vec F_1 = \vec F_2 = -I a k \frac{a}{2}\hat z$$

Show 5

$$= -ka^2\frac{I}{2}\hat z$$

$$\vec F_{total} = -ka^2 I\hat z$$ calculate 5

EM

(A4) (1) crossed fields: el. and magn. forces compensate each other

4 $E = vB, \quad v = \frac{E}{B}$

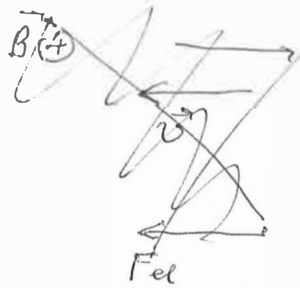
(2) magn field → circular motion

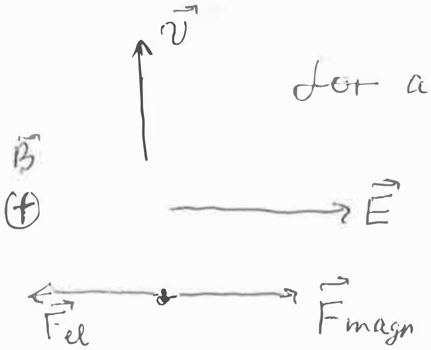
4 $\frac{mv^2}{R} = evB$

$$\frac{e}{m} = \frac{v}{RB} = \frac{E}{RB^2} = \frac{2.2\times10^3}{0.05\cdot(5\times10^{-4})^2} = 1.76\times10^{11}\,\frac{C}{kg}$$

solve 3 calculate 4

For part (a)





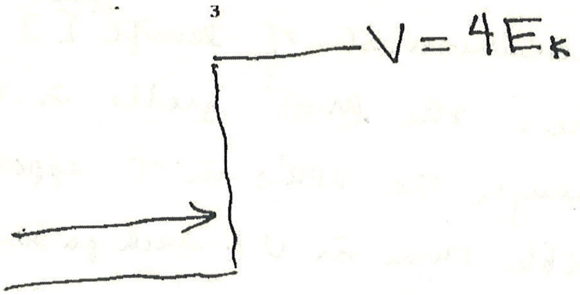
for a negative charge

10

or similar

19 M A1

Estimate the quantum mechanical penetration of a small particle of dust of radius 10^{-9} m and density $\rho = 10^5$ kg/m^3, moving at velocity v = 10 m/s into a barrier with 4 times the particle's kinetic energy. Assume that penetration occurs only for a probability greater than 1/e, and that the volume of a sphere is $\frac{4}{3}\pi R^3$.



Set up 5

$$\psi = Be^{-kx} \qquad \psi^*\psi = B^2 e^{-2kx} = \frac{B^2}{e} \qquad \boxed{\ln\left(\frac{1}{e}\right) = -1}$$

Solve 5

$$-2kx = \ln\left(\frac{1}{e}\right) \qquad \boxed{m = \rho\,\text{Vol.} = 4.2\times10^{-22}\ \text{kg}}$$

mass 4

$$x = \frac{\ln(e)}{2\frac{\sqrt{2m\,3E_k}}{\hbar}}$$

$$\boxed{V - E = 4E_k - E_k = 3E_k}$$

$$\boxed{E_k = \frac{1}{2}mv^2 = \frac{10^5\,\text{kg/m}^3\,\frac{4}{3}\pi R^3 v^2}{2} = 2.09\times10^{-20}\,\text{J}}$$

Ek 5

Calculate 6

$$= \frac{1.05\times10^{-34}\ \text{Js}}{2\sqrt{6\times2.09\times10^{-20}\times4.2\times10^{-22}}}$$

$$= 7.2\times10^{-15}\ \text{m}$$

QM (A2)

S=0,1 -- ~~3~~ 4 pt

) (~~15~~ pt (a) Eigenvalues of $\boldsymbol{S}^2$ are $\hbar^2 S(S+1)$ where $S = 0, 1$, i.e.,

$$\{\hat{S}^2\} = \hbar^2 S(S+1) = \hbar^2\{0, 2, 2, 2\}.$$

state -- ~~2~~ 3 pt

degenercy right -- ~~2~~ 4 pt

~~Eigenvectors are:~~

~~each eigen vector, 2pt~~

$$S = 0: \quad \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle - \left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle\right)$$

$$S = 1: \quad \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle + \left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle\right), \quad \left|-\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle, \quad \left|\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle$$

-- ~~2~~ 3 pt

7 ~~5~~pt (b) $\hat{s}_1 \cdot \hat{s}_2 = \hat{S}^2/2 - 3\hbar^2/4 = \hbar^2(S(S+1)/2 - 3/4)$. Therefore

$$\{\hat{V}_1\} = \hbar^2(a - 2bS(S+1) - 3b) = \hbar^2\{a - 3b, a + b, a + b, a + b\}.$$

final answer -- ~~3~~ 4 pt

commute, -- ~~2~~ 3 pt

7 ~~5~~pt (c) $\hat{S}_z = \hat{s}_{1z} + \hat{s}_{2z}$ with eigenvalues $S_z = \hbar\{0, 1, -1\}$ commutes with $\hat{S}^2$. Therefore,

$$\{\hat{V}_2\} = a\hbar S_z + b\hbar^2(2S(S+1) - 3) = \hbar^2\{-3b, b, a + b, -a + b\}.$$

final answer-- ~~3~~ 4 pt

3

5pt (a) ~~For one-dimensional Hamiltonian H_l:~~

5pt

$$\frac{\partial E_{nl}}{\partial l} = \overline{\frac{\partial \hat{H}_l}{\partial l}} = \overline{\frac{\hbar^2(2l+1)}{2mr^2}} > 0,$$

~~i.e., E_{nl} increases with increasing l.~~

5pt (b) ~~Since the energy of the level decreases with decreasing l there must be at least l levels below a level with angular momentum l. Since there are only $N-1$ levels below the N-th, $l_{\max} = N - 1$.~~

5pt

5pt (c) ~~$n_r = 0$. Otherwise, below this level, in addition to $N-1$ levels corresponding to $l = 0, 1, \ldots, l_{\max}$, there will be also $n_r' = n_r - 1$ level.~~

5pt

10pt (d) ~~The maximum degeneracy can be achieved if all levels with $l = 0, 1, \ldots, l_{\max}$ are degenerate. This means~~

condition for maximum degeneracy achieved -- 5pt

QM (A3)

$$\psi(\vec r)=\frac{1}{(2\pi\hbar)^{3/2}}\int e^{i\vec p\cdot\vec r/\hbar}\,\phi(\vec p)\,d\vec p$$

inv transform
5

$$\phi(\vec p)=\frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i\vec p\cdot\vec r/\hbar}\,\psi(\vec r)\,d\vec r$$

set up 3

$$\int\phi^*(\vec p)\,\phi(\vec p)\,d\vec p=\frac{1}{(2\pi\hbar)^3}\int e^{-i\vec p\cdot\vec r/\hbar+i\vec p\cdot\vec r'/\hbar}\,\psi^*(\vec r')\,\psi(\vec r')\,d\vec r\,d\vec r'\,d\vec p$$

10 integrate in $\vec p$ first and use $\frac{1}{2\pi}\int e^{ip(x'-x)}dp=\delta(x'-x)$

$\delta(ax)=\frac{1}{a}\delta(x)$

↑

similar in 3d

$\frac{1}{(2\pi)^3}\int e^{i\vec k\cdot(\vec r-\vec r')}d\vec k$

$=\delta(\vec r-\vec r')$

$\delta(a\vec r)=\frac{1}{a^3}\delta(\vec r)$

3 $$=\int\delta(\vec r-\vec r')\,\psi^*(\vec r')\,\psi(\vec r)\,d\vec r\,d\vec r'$$

4 $$=\int\psi^*(\vec r)\,\psi(\vec r)\,d\vec r=1$$

QM ⓐ A4

$$\psi_{1s} = \frac{1}{\sqrt{4\pi}} R_{10}(r)$$

3 $$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

Set up 7 $$\langle r \rangle = \int R_{10}^2(r)\, r^3\, dr = \frac{4}{a_0^3} \int e^{-2r/a_0} r^3\, dr = \frac{4}{a_0^3} \left(\frac{a_0}{2}\right)^4 \int e^{-x} x^3\, dx$$

Calculate 7

$$x = \frac{2r}{a_0}$$

$$\langle r \rangle = \frac{a_0}{4}\, 3! = \frac{3}{2} a_0$$

8 $$\langle z \rangle = \frac{1}{2} \overset{\nearrow 0}{\int_0^{\pi} \cos\theta \sin\theta\, d\theta} \int R_{10}^2(r)\, r^3\, dr = 0$$

or simply $\langle z \rangle = 0$ because the integrand is the odd function of z

QM (B1)

~~$\langle 0|\hat{V}|1^+\rangle = -\varepsilon \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} 1 \cos\phi\cos\phi = -\varepsilon/\sqrt{2}\,,$~~ 1 pt

~~$\langle 2^+|\hat{V}|1^+\rangle = -\varepsilon \int_0^{2\pi} \frac{d\phi}{\pi} \cos 2\phi \cos\phi\cos\phi = -\varepsilon/2\,.$~~ 1 pt

~~Thus~~

~~$\Delta E^{(2)}_{1^+} = \frac{|\langle 0|\hat{V}|1^+\rangle|^2}{E_1 - 0} + \frac{|\langle 2^+|\hat{V}|1^+\rangle|^2}{E_1 - E_2} = \frac{2I\varepsilon^2}{\hbar^2}\left(\frac{1/2}{1-0} + \frac{1/4}{1-4}\right) = \frac{5I\varepsilon^2}{6\hbar^2}$~~ 1 pt

~~Similarly, for the 1^- state:~~

~~$\langle 0|\hat{V}|1^-\rangle = -\varepsilon \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} \cos\phi\sin\phi = 0$~~ 1 pt

~~$\langle 2^-|\hat{V}|1^-\rangle = -\varepsilon \int_0^{2\pi} \frac{d\phi}{\pi} \sin 2\phi \cos\phi\sin\phi = \varepsilon/2\,.$~~ 1 pt

~~Thus~~

~~$\Delta E^{(2)}_{1^-} = +\frac{|\langle 2^+|\hat{V}|1^+\rangle|^2}{E_1 - E_2} = \frac{2I\varepsilon^2}{\hbar^2}\frac{1/4}{1-4} = -\frac{I\varepsilon^2}{6\hbar^2}$~~ 1 pt

Hard -3 (5pt) use $\hat{L}_+$ $\hat{L}_-$ 2 pts $\bar{L}_+$ $\bar{L}_-$ 2pts

(a) Since L_+ has nonzero matrix elements only between states with different values of M (different by 1), the expectation value $\overline{\hat{L}_+} = 0$. Similarly, $\overline{\hat{L}_-} = 0$ and, since $\hat{L}_{x,y}$ are linear combinations of $L_{+,-}$:

$\bar{L}_x, \bar{L}_y \Rightarrow 1$ pt

$\overline{\hat{L}_x} = \overline{\hat{L}_y} = 0.$

(b) (5pt) ↓2 pt ↓3 pt

$$\overline{\hat{L}_x^2 + \hat{L}_y^2} = \overline{\hat{\mathbf{L}}^2 - \hat{L}_z^2} = \hbar^2(L(L+1) - M^2)$$

(c) (5pt) Since $\hat{L}_+^2$ has nonzero matrix elements only between states with different values of M (different by 2), the expectation value $\overline{\hat{L}_+^2} = 0$. I.e., 2pt

$0 = \overline{(\hat{L}_x + i\hat{L}_y)^2} = \overline{\hat{L}_x^2 - \hat{L}_y^2} + i\overline{(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x)}\,.$ 1pt

Which means

$$\overline{\hat{L}_x^2 - \hat{L}_y^2} = \overline{\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x} = 0\,.$$ 2pt

(d) (5pt)

$$\hat{L}_{z'} = \hat{L}_z\cos\theta + (\hat{L}_x\cos\phi + \hat{L}_y\sin\phi)\sin\theta\,.$$ 3pt

B1 p.2

$$\overline{\hat{L}_{z'}} = \hbar M \cos\theta .$$ 2pt

(e) (5pt)

$$\overline{\hat{L}^2_{z'}} = \hbar^2 \left(M^2 \cos^2\theta + \frac{1}{2}\left(L(L+1) - M^2\right)\sin^2\theta \right)$$

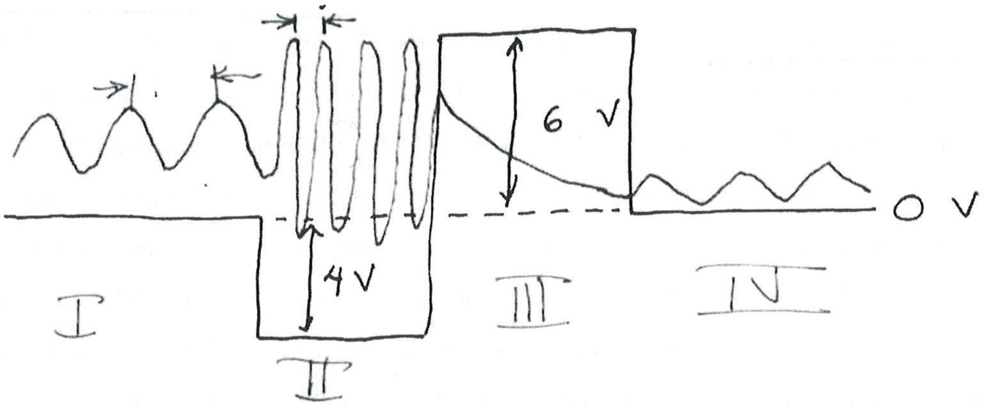
↑ 3pt

$$= \hbar^2 \left(M^2 + \frac{1}{2}\left(L(L+1) - 3M^2\right)\sin^2\theta \right)$$

↑ 2pt

QM $\boxed{B2}$

Consider the potential shown. Draw in as much detail as you can the wave function of a particle at energy 4 eV and explain the main characteristics of your wave function.



(1) $\lambda_{II}\sqrt{2} = \lambda_{I}$. The kinetic energy doubled $E = h\nu = \frac{p^2}{2m}$

5 $2E_{I} = E_{II}$ so $P_{I}\sqrt{2} = P_{II}$ but $p = \frac{h}{\lambda}$ so 3 $\lambda_{II} = \frac{h}{P_{I}\sqrt{2}}$

5 (2) the amplitude in II increase, as the wave function has a higher probability to be found in II.

5 (3) exponential decay. Particle is classically forbidden from III.

5 (4) $\lambda_{I} \equiv \lambda_{IV}$

2? (5) wave function is continuous and single valued

$\boxed{B3}$ QM

(a) $\psi(x) = A(a^2 - x^2)$

Setup 2 $A^2 \int_{-a}^{a} (a^2-x^2)^2 dx = A^2 \int (a^4 - 2a^2x^2 + x^4)\, dx$

Calculate 4

$$= A^2\left(a^4 \cdot 2a - 2a^2 \cdot \frac{2}{3}a^3 + 2\frac{a^5}{5}\right) = A^2 a^5\left(2 - \frac{4}{3} + \frac{2}{5}\right) = \frac{16}{15}A^2a^5 = 1$$

$$A = \frac{\sqrt{15}}{4} a^{-5/2}$$

(b) measurement outcomes — energy eigenvalues

5 $$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2, \quad L = 2a \to E_n = \frac{\hbar^2}{8m}\left(\frac{\pi n}{a}\right)^2$$

(c) For the ground state the wavefunction is ($x \to x+a$, $L = 2a$) from $\left(\frac{2}{L}\right)^{1/2}\sin\frac{\pi x}{L}$

3 $$\psi_1(x) = \left(\frac{2}{2a}\right)^{1/2} \sin\frac{\pi}{2a}(x+a) = \frac{1}{a^{1/2}}\cos\frac{\pi x}{2a}$$

probability amplitude $\int \psi(x)\psi_1(x)\,dx$

Setup 2 $$\frac{A}{\sqrt{a}}\int_{-a}^{a}(a^2-x^2)\cos\frac{\pi x}{2a}\,dx = \frac{A}{\sqrt{a}}\left[\overset{\nearrow 0}{(a^2-x^2)\frac{2a}{\pi}\sin\frac{\pi x}{2a}\Big|_{-a}^{a}}\right.$$

$$+ \frac{2a}{\pi}\int_{-a}^{a} 2x\sin\frac{\pi x}{2a}\,dx = \frac{A}{\sqrt{a}}\frac{4a}{\pi}\left(\overset{\nearrow 0}{-x\frac{2a}{\pi}\cos\frac{\pi x}{2a}\Big|_{-a}^{a}} + \right.$$

Calculate 7

$$\left.\frac{2a}{\pi}\int\cos\frac{\pi x}{2a}\,dx\right) = \frac{4A\sqrt{a}}{\pi}\left(\frac{2a}{\pi}\right)^2\sin\frac{\pi x}{2a}\Big|_{-a}^{a} = \frac{16Aa^{5/2}}{\pi^3}\cdot 2$$

$$= \frac{8\sqrt{15}}{\pi^3}$$

$$P = \frac{15 \cdot 64}{\pi^6} = 0.99855$$

(d) meaning that $\psi(x)$

2 is close to $\psi_1(x)$ (the w.f. of the ground state)

By a similar rationale,

$$\langle p \rangle = 0 \tag{8}$$

<p> 3pt

5pt (c) The state is a superposition of two eigenstates which correspond to two possible measurement outcomes for total energy:

$$|0\rangle \rightarrow E = \hbar\omega/2 \tag{9}$$

$$|2\rangle \rightarrow E = 5\hbar\omega/2 \tag{10}$$

(9) and (10) 2pt

The probability of each potential is the square magnitude of the projection onto the corresponding eigenstate:

$$E = \hbar\omega/2: \quad P = |\langle 0|\Psi(T)\rangle|^2 = 1/5 \tag{11}$$

$$E = 5\hbar\omega/2: \quad P = |\langle 2|\Psi(T)\rangle|^2 = 4/5 \tag{12}$$

(11) and (12) 3 pt

5pt (d) The measurement at $t = T$ collapses the wave function to $|\Psi(t)\rangle = e^{-i5\omega t/2}|2\rangle$ for $t > T$. Therefore, all subsequent measurements of total energy will yield $E = 5\hbar\omega/2$.

no partial, just 5 pt

2 QM (B4)

5pt (a) $|\psi\rangle = \frac{1}{\sqrt{3}}|j, m+1\rangle + \frac{1}{\sqrt{3}}|j, m\rangle + \frac{1}{\sqrt{3}}|j, m-1\rangle$ 3pt

The possible values of J_z are

$$\hbar(m, m \pm 1) \qquad \text{each with probability} \quad \frac{1}{3} \tag{1}$$

1pt

1pt

5pt (b) *Find the expectation value of measurement J_z in this state, and uncertainty of its measurement.*

$$\langle J_z \rangle = \langle \psi | J_z | \psi \rangle = \frac{\hbar}{3}[(m-1) + m + (m+1)] = \hbar m \quad \text{2pt} \qquad (2)$$

$$\Delta J_z^2 = \langle \psi | J_z^2 | \psi \rangle - \langle \psi | J_z | \psi \rangle^2 = \frac{\hbar^2}{3}[(m-1)^2 + m^2 + (m+1)^2] - (\hbar m)^2 = \frac{2}{3}\hbar^2 \qquad (3)$$

5 ~~3~~pt

10pt (c) *Using properties of $J_\pm = J_x \pm iJ_y$ operators show that the expectation value of J_y in this state is zero. Find the expectation value of J_x in this state, simplify it in the case $j \pm m \gg 1$*

$$\langle J_y \rangle = \frac{1}{2i} \langle \psi | J_+ - J_- | \psi \rangle = \frac{1}{2i}(\langle \psi | J_+ | \psi \rangle - \langle \psi | J_+ | \psi \rangle^*) = Im \, \langle \psi | J_+ | \psi \rangle \qquad (4)$$

5 5 ~~3~~pt

Since the state is purely real, and the matrix coefficients of J_+-operator are real, the imaginary part of this expression is zero, $Im \, \langle \psi | J_+ | \psi \rangle = 0$.

$$\langle J_x \rangle = \frac{1}{2} \langle \psi | J_+ + J_- | \psi \rangle = \frac{1}{2}(\langle \psi | J_+ | \psi \rangle + \langle \psi | J_+ | \psi \rangle^*) = Re \, \langle \psi | J_+ | \psi \rangle \qquad (5)$$

5 ~~3~~pt

The non-zero contributions to this average come from terms

$$\begin{aligned}\langle J_x \rangle &= \frac{\hbar}{3} \langle j, m+1 | J_+ | j, m \rangle + \frac{\hbar}{3} \langle j, m | J_+ | j, m-1 \rangle \\ &= \frac{\hbar}{3}\left(\sqrt{(j-m)(j+m+1)} + \sqrt{(j-m+1)(j+m)}\right) \\ &\approx \hbar \frac{2}{3}\sqrt{j^2 - m^2}\end{aligned}$$

5 ~~4~~pt

5pt (d) ~~*Using previous two steps and the general uncertainty principle, what is the minimal uncertainty in measuring J_y in this state?*~~

$$\Delta J_y \Delta J_z \geq \frac{1}{2} | \langle [J_y, J_z] \rangle | = \frac{\hbar}{2} | \langle J_x \rangle | \quad \text{3pt} \qquad (6)$$

$$\Delta J_y \geq \frac{\hbar}{2} \frac{| \langle J_x \rangle |}{\Delta J_z} = \frac{\hbar}{2}\sqrt{\frac{2}{3}}\sqrt{j^2 - m^2} = \hbar\sqrt{\frac{j^2 - m^2}{6}} \qquad (7)$$

2pt