EM problems

A1.. An infinite plane of a uniform surface charge density σ_0 is placed at distance $z = h$ above the surface of a half-space grounded metal.

(a) Find the potential and the electric field in the whole space.

(b) Find the induced surface charge density on a metal surface.

Solution:

(a) The solution can be obtained using a method of images. An image charge plane is located at $z = -h$ below the surface of a metal and has charge $-\sigma_0$ per unit area. The electric file produced by the two planes of charge is uniform between the planes and is equal to $\mathbf{E} = -\frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{z}}$. Above the plane of charge, $z > h$, the electric field is zero. Since $\mathbf{E} = -\nabla\Phi$, we find that between the planes the potential is $\Phi(z) = \frac{\sigma_0}{\varepsilon_0}z + C$, where constant C must be equal to zero in order to have zero potential on the grounded metal surface. Above the plane of charge the potential is constant and is equal to $\Phi(z) = \frac{\sigma_0}{\varepsilon_0}h$. Thus,

8 points

9 points.

$$\Phi(z) = 0; \quad \mathbf{E} = 0; \quad z < 0$$

$$\Phi(z) = \frac{\sigma_0}{\varepsilon_0}z; \quad \mathbf{E} = -\frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{z}}; \quad 0 < z < h, \tag{1}$$

$$\Phi(z) = \frac{\sigma_0}{\varepsilon_0}h; \quad \mathbf{E} = 0; \quad z > h$$

at $z = 0$ and $z = h$ the potential is continuous, while the electric field is discontinuous.

(b) The induced surface charge density σ on a metal surface is given by

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}, \tag{2}$$

$\mathbf{E}_{above}$ is the electric field above the metal surface, $\mathbf{E}_{below}$ is the electric field below the metal surface, and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ is the normal to the surface. From eqs.(1) and (2) we find that $\sigma = -\sigma_0$.

8 points.

B1. A flat surface z=0 of a semiinfinite linear dielectric material of uniform dielectric permittivity ε is affected by an external non-uniform electric field which is directed perpendicular to the surface and whose magnitude in the absence of the dielectric is $E^{ext}(\mathbf{r})$. There are no free charges in the dielectric.

(1) Show that the volume polarization charge density in the dielectric is zero;

(2) Show that the normal component of the polarization-induced electric field $E_z^P(\mathbf{r})$ near the surface inside the dielectric is given by $E_z^P(\mathbf{r}) = -\frac{\sigma_P(\mathbf{r})}{2\varepsilon_0}$ where $\sigma_P(\mathbf{r})$ is the induced surface polarization charge density.

(3) Express $\sigma_P(\mathbf{r})$ is terms of the total electric field at the surface;

(4) Express $\sigma_P(\mathbf{r})$ in terms of $E^{ext}(\mathbf{r})$.

Solution:

6 points

(1) For a linear dielectric **P** and **E** are related by $\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}$, where **E** is the total electric field in the dielectric. Therefore 1 point

$$\rho_P = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left[(\varepsilon-\varepsilon_0)\mathbf{E}\right]$$ 2 pts.

If the dielectric is *uniform* then $(\varepsilon - \varepsilon_0)$ is constant and thus can be taken out of the derivative:

$$\rho_P = -(\varepsilon-\varepsilon_0)\nabla\cdot\mathbf{E} = -\frac{(\varepsilon-\varepsilon_0)}{\varepsilon_0}\left(\rho_{free}+\rho_P\right)$$ 2 pts.

$$\Rightarrow \rho_P = \left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_{free}$$ 1 pt.

Thus, if there are no free charges and the dielectric is linear and uniform, the volume polarization charge density is zero.

6 points

(2) Electrostatic boundary conditions state that the normal component of the electric field experiences a step of $\frac{\sigma(\mathbf{r})}{\varepsilon_0}$ when crossing the surface. (1 point) Since there are no free charges the surface charge density $\sigma(\mathbf{r})$ is equal to the surface polarization charge density $\sigma_P(\mathbf{r})$. (1 point) Since the surface is a plane and there are no polarization charges in the bulk, by symmetry the normal component of the polarization-induced electric field above and below the surface must by equal in magnitude but pointing in the opposite directions. (2 points) Therefore $E_z^P(\mathbf{r}) = -\frac{\sigma_P(\mathbf{r})}{2\varepsilon_0}$. (1 point)

$E^P_{2n} - E^P_{in} = \frac{\sigma_P}{\varepsilon_0}$ (outside) (inside)

6 points

(3) The surface polarization charge density is given by

$$\sigma_P(\mathbf{r}) = \mathbf{P}\cdot\mathbf{n}$$ 1 point

where **n** is the normal to the surface. Polarization **P** is a linear function of electric field so that

$$\mathbf{P} = (\varepsilon-\varepsilon_0)\mathbf{E}.$$ Total field. 2 points

Therefore we find:

$$\sigma_P(\mathbf{r}) = (\varepsilon-\varepsilon_0)E_z,$$ 2 points

where E_z is the normal component of the electric field inside the dielectric near its surface.

7 points

(4) The total electric field is the sum of the external field and the field produced by polarization charge $\mathbf{E} = \mathbf{E}^P + \mathbf{E}^{ext}$. Therefore we have:

1 point

2 points 2 point

$$\left[\sigma_P = (\varepsilon-\varepsilon_0)E_z = (\varepsilon-\varepsilon_0)\left(E_z^P + E_z^{ext}\right)\right] = \left[(\varepsilon-\varepsilon_0)\left(-\frac{\sigma_P}{2\varepsilon_0} + E_z^{ext}\right)\right]$$

Solving this equation with respect to σ_P we find:

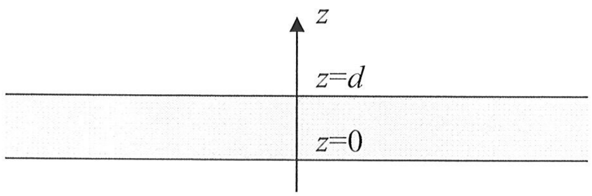
$$\left.\sigma_P(\mathbf{r}) = 2\varepsilon_0 \frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0} E_z^{ext}(\mathbf{r})\right]$$

2 points.

A2. A positive charge is distributed over a thin metal plate of infinite area and thickness d in such a way that there is charge Q per area A of the plate. Assuming that the bottom surface of the plate lies at $z = 0$ and the top surface lies at $z = d$,

(a) Find the distribution of charge across the plate (along the z direction) and write the expression for the volume charge density in all space;

(b) Find and sketch the electric field and the electrostatic potential along the z direction.



Solution:

10 points

(a) Since the plate is infinite the charge is distributed uniformly over the area of the plate in such a way that produces zero electric field inside the metal. (3 points) Since the charge density inside the metal is zero, the only way to have zero electric field inside the metal is to have a uniform surface charge density $\sigma = Q/2A$ on the top and bottom surfaces. These surface charge densities produce uniform electric fields pointing in the opposite directions inside the plate making the net field equal to zero. (5 points) In this case the volume chare density is given by

$$\rho(\mathbf{r}) = \frac{Q}{2A}\delta(z) + \frac{Q}{2A}\delta(z-d).$$ 2 points

15 points

(b) The electric field outside the plate is the same as the field produced by uniform plane charge of surface density $\sigma = Q/A$, i.e.

(Can also add the field from each plate)

$$\mathbf{E} = \frac{Q}{2A\varepsilon_0}\hat{\mathbf{z}}, \quad z > d;$$ 2 pts $\frac{Q}{4A\varepsilon_0} + \frac{Q}{4A\varepsilon_0}\hat{z}$

$$\mathbf{E} = -\frac{Q}{2A\varepsilon_0}\hat{\mathbf{z}}, \quad z < 0.$$ 2 points

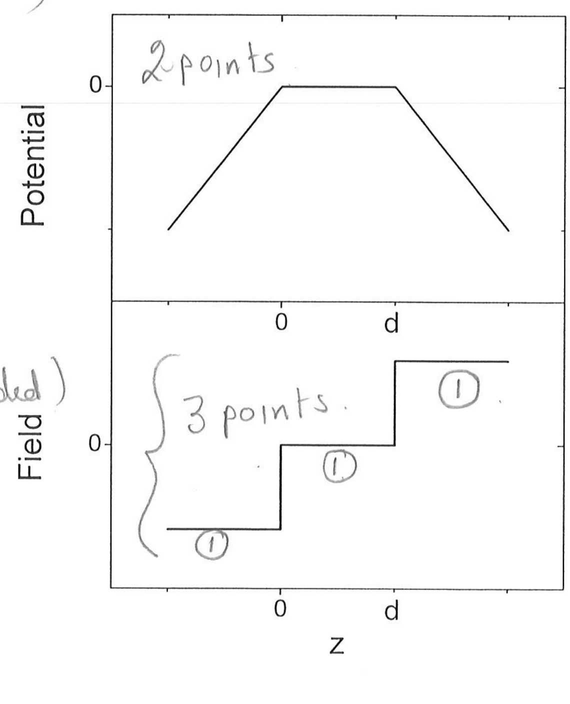
The electrostatic potential is determined from $\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial z}$. Assuming that the potential is zero on the metal plate we find

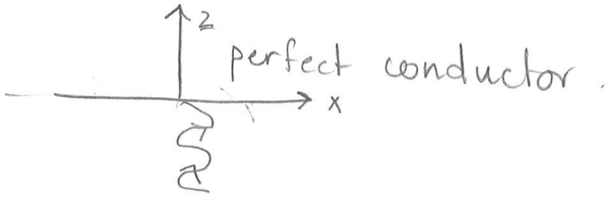
Can set zero anywhere (it's not grounded)

$$\Phi = -\frac{Q}{2A\varepsilon_0}(z-d), \quad z > d;$$ 3 points

zero of potential

$$\Phi = \frac{Q}{2A\varepsilon_0}z, \quad z < 0.$$ 3 points





B2.. A linearly polarized electromagnetic wave plane wave of frequency ω, traveling in vacuum in the z direction and polarized in the x direction, is reflected from a perfect conductor (i.e., a conductor with infinite conductivity σ) whose surface lies in the x-y plane. Write expressions for the incident, transmitted, and reflected waves and using the boundary conditions for the electric and magnetic fields show that the reflection coefficient is equal to 1.

Solution:

The electric and magnetic fields of the incident and reflected wave can be expressed as follows: (6 points.)

$$\mathbf{E}_i(z,t) = E_{0i} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}, \quad \text{— 1 points.}$$

$$\mathbf{B}_i(z,t) = \frac{1}{c} E_{0i} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}, \quad \text{— 1 point.}$$

$$\mathbf{E}_r(z,t) = E_{0r} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}, \quad \text{— 2 points.}$$

$$\mathbf{B}_r(z,t) = -\frac{1}{c} E_{0r} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}, \quad \text{— 2 points}$$

where $k_1 = \dfrac{\omega}{c}$. The electric and magnetic fields of the transmitted wave are (6 points.)

$$\mathbf{E}_t(z,t) = E_{0t} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}, \quad \text{2 points.}$$

$$\mathbf{B}_t(z,t) = \frac{k_2}{\omega} E_{0t} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}, \quad \text{2 points.}$$

where k_2 is a complex wave vector given by $k_2 = \dfrac{\omega}{c}\sqrt{\dfrac{\varepsilon}{\varepsilon_0}}$ and $\varepsilon = \varepsilon_0 + i\dfrac{\sigma}{\omega}$. (2 points)

(3 points) The boundary condition $\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$ results in

$$E_{0i} + E_{0r} = E_{0t}$$

(4 points) and the boundary condition $\mathbf{B}_1^{\parallel} = \mathbf{B}_2^{\parallel}$ leads to

$$\frac{1}{c}\left(E_{0i} - E_{0r}\right) = \frac{k_2}{\omega} E_{0t} \quad \text{(2 points)} \quad \text{or} \quad E_{0i} - E_{0r} = \beta E_{0t}, \quad \text{(2 pts.)}$$

where $\beta = \dfrac{c}{\omega} k_2$.

(3pts) Solving these equations we find:

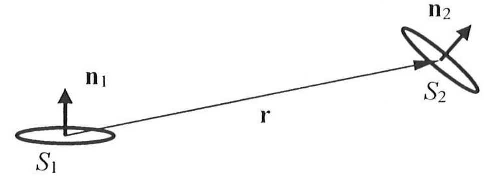
$$E_{0r} = \frac{1-\beta}{1+\beta} E_{0i}, \quad E_{0t} = \frac{2}{1+\beta} E_{0i}.$$

(3pts.) For a perfect conductor $\sigma = \infty$, $k_2 = \infty$ and hence $\beta = \infty$. Therefore,

$$E_{0r} = -E_{0i}, \quad E_{0t} = 0.$$

This implies that $R = \left|\dfrac{E_{0r}}{E_{0i}}\right|^2 = 1$.

B3. Two small planar loops of wire of area S_1 and S_2 have normal unit vectors to their planes, $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively, and are separated by radius vector $\mathbf{r}$, large compared to the size of the loops.



(a) Find the mutual inductance of the system.

(b) Assuming that $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel find the angle between $\mathbf{r}$ and $\mathbf{n}_1$ ($\mathbf{n}_2$) at which the mutual inductance is zero.

(c) Sketch magnetic field lines with respect to the current loops, explaining the result (b).

Solution:

(a) Since the separation between the loops is large we can use the dipole approximation to for the magnetic field. Using a unit vector $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, the magnetic field due to loop S_1 at a position of loop S_2 is (eq.5.56 in the textbook): (formula sheet)

1

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}_1) - \mathbf{m}_1}{r^3}\right]. \tag{5.1}$$

3 Since for a planar loop $\mathbf{m}_1 = IS_1\mathbf{n}_1$, where I is the current flowing the wire, we obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} IS_1\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{n}_1) - \mathbf{n}_1}{r^3}\right]. \tag{5.2}$$

5 The flux crossing loop S_2 is (we assume $\bar{B}$ is constant since $\bar{r} >>$ diameter of loop.)

$$F = S_2\mathbf{B}\cdot\mathbf{n}_2 = \frac{\mu_0}{4\pi} IS_1S_2\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{n}_1)(\hat{\mathbf{r}}\cdot\mathbf{n}_2) - \mathbf{n}_1\cdot\mathbf{n}_2}{r^3}\right]. \tag{5.3}$$

2 The mutual inductance is therefore

$$M_{12} = \frac{F}{I} = \frac{\mu_0}{4\pi} S_1S_2\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{n}_1)(\hat{\mathbf{r}}\cdot\mathbf{n}_2) - \mathbf{n}_1\cdot\mathbf{n}_2}{r^3}\right]. \tag{5.4}$$

2 (b) The mutual inductance becomes equal to zero when

$$3(\hat{\mathbf{r}}\cdot\mathbf{n}_1)(\hat{\mathbf{r}}\cdot\mathbf{n}_2) = \mathbf{n}_1\cdot\mathbf{n}_2. \tag{5.5}$$

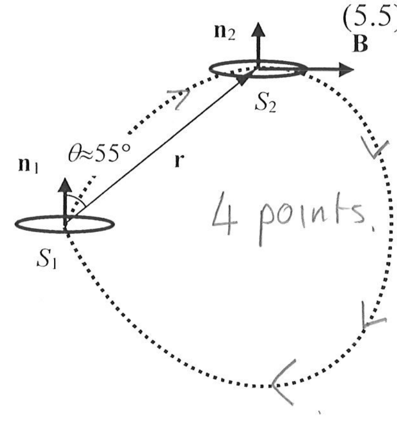
4 If $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel then $\mathbf{n}_1\cdot\mathbf{n}_2 = 1$ and we find

$$\cos^2\theta = 1/3, \tag{5.6}$$

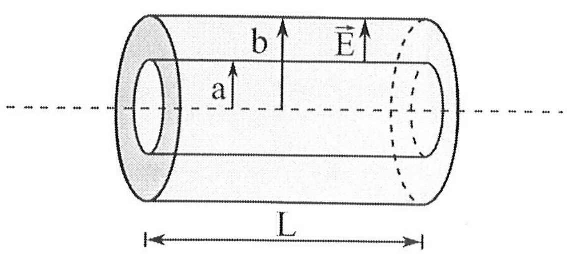
where θ is the angle between $\mathbf{r}$ and $\mathbf{n}_1$ ($\mathbf{n}_2$) so that $\cos\theta = (\hat{\mathbf{r}}\cdot\mathbf{n}_1) = (\hat{\mathbf{r}}\cdot\mathbf{n}_2)$. This gives

$$\theta \approx \pm 54.7°, \pm 125.3°. \tag{5.7}$$

4 At these angles the plane of loop S_2 lies parallel to the magnetic field lines produced by loop S_1 and consequently the flux is zero, as is seen from the figure.



B4. Most materials are neither perfect conductors nor perfect dielectrics. Many capacitors with dielectrics will "leak" a small current. Consider a cylindrical coaxial capacitor with a leakage current between the two long electrodes of radii *a* and *b*, with *b* > *a*, separated by an unknown material of conductivity σ and permittivity close to ε_0. They are maintained at an electric potential *V* with a current *I* flowing from one to another in a length *L*. Find the conductivity σ.



Solution:

Let us start by calculating the electric field between the cylinders at a radius s, with s between the radii *a* and *b*. We consider λ the charge per unit length of the inner cylinder. Gauss's law states

10 points

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0}. \quad \longrightarrow \text{2 points}$$

The left side of the equation is $E2\pi sL$, while the enclosed charge is simply λL. We therefore have

$$E2\pi sL = \frac{\lambda L}{\varepsilon_0}, \quad \text{6 points}$$

so that the electric field is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}} \quad \text{(important) 2 points}$$

6 points

According to Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ and therefore

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{\lambda}{2\pi\varepsilon_0 s}2\pi sL = \frac{\sigma\lambda L}{\varepsilon_0}.$$

4 points. Integral over curved surface.

5 points

The electric potential is

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \frac{\lambda}{2\pi\varepsilon_0 s}ds = -\frac{\lambda}{2\pi\varepsilon_0}\ln s\Big|_b^a = \frac{\lambda}{2\pi\varepsilon_0}(\ln b - \ln a) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}.$$

4 points.

3 points

From the last two relations we find: (Substitute for λ from I)

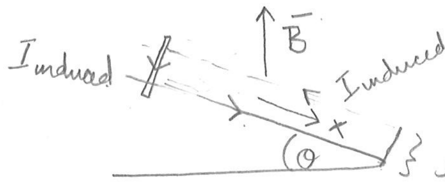
$$V = \frac{\frac{\varepsilon_0 I}{\sigma L}}{2\pi\varepsilon_0}\ln\frac{b}{a} = \frac{I}{2\pi\sigma L}\ln\frac{b}{a}$$

1 point

and therefore

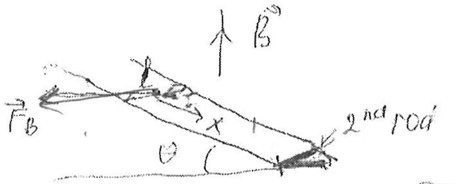
$$\sigma = \frac{I}{2\pi LV}\ln\frac{b}{a}.$$

A3. A conducting rod of mass m and resistance R is free to slide without friction along two parallel rails of negligible resistance. The rails are separated by a distance *l* and inclined at an angle θ to the horizontal. Another conducting rod of negligible resistance is placed at the base of the rails making a closed circuit. There is a magnetic field *B* directed upward. Find the terminal speed of the rod. (Assume that the rails are long enough for this).



EM (A3)

2 points [As the rod slides down, there is a decrease in magnetic flux.



Change of flux in the closed circuit

θ is angle between $\bar{B}$ + $d\bar{a}$

$d\Phi = Bl\,dx\cos\theta$] 4 points

[Therefore emf, according to the Faraday law is

$\left|\frac{d\Phi}{dt}\right| = Blv\cos\theta \qquad I = \frac{Blv\cos\theta}{R}$] 4 points.

[According to the Lenz law, the induced current in the rod is such that the magnetic force F_B prevents rod from sliding. [Therefore it is directed to the left (The induced current is ccw looking down)] 5 points

6 points [Equation of motion along x direction

$m\ddot{x} = mg\sin\theta - F_B\cos\theta$ (mg sin θ: down the slope; $F_B\cos\theta$: component of F_B along − x direction)

[$F_B = BIl = \frac{(Bl)^2 v\cos\theta}{R}$ This is horizontal to the left]]

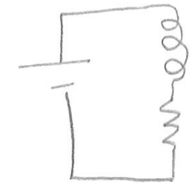
4 points [when the rod achieves ~~perman~~ terminal velocity,

$\ddot{x} = 0 \rightarrow mg\sin\theta = \frac{(Bl)^2 v_t\cos^2\theta}{R}$

$v_t = \frac{mgR\sin\theta}{(Bl)^2\cos^2\theta}$]

A4. A large electromagnet has an inductance 40 H and a resistance 10 Ω. It is connected to a dc power source of 250V. Find the time for the current to reach 10 A.

EM (A4)



$$\left[\mathcal{E} = L\frac{dI}{dt} + IR\right]$$ 5 points

Solution of homogeneous eq. ($L\frac{dI}{dt} + IR = 0$, $\frac{dI}{dt} = -\frac{R}{L}I$)

$$I = ce^{-\frac{R}{L}t}$$

general solution of inhomog eq.

$$I = ce^{-\frac{R}{L}t} + \frac{\mathcal{E}}{R}$$

initial condition $I(0) = 0$, therefore $\Rightarrow$ $c = -\frac{\mathcal{E}}{R}$

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

7 points

Or get this from memory?

Solve for t:

$$\frac{R}{L}t = -\ln\left(1 - \frac{RI}{\mathcal{E}}\right)$$ 7 points

$$t = -\frac{L}{R}\ln\left(1 - \frac{RI}{\mathcal{E}}\right) = -\frac{40}{10}\ln\left(1 - \frac{10 \cdot 10}{250}\right)$$

$$= 2.04\,s$$

6 points

QM (A1)

(a) $p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240\,eV\cdot nm}{200\,nm\cdot c} = 6.20\,\frac{eV}{c} = \frac{6.20\cdot 1.6\times 10^{-19}}{3\times 10^{8}}\,\frac{kg\cdot m}{s}$

$= 3.31\times 10^{-27}\,\frac{kg\cdot m}{s}$

6pts 6pts

(b) $p_{ph} = p_{el} + p_{pan}$ 3pts $\xrightarrow{p_{ph}}$ $\xleftarrow{p_{el}}$

$p_{pan} = p_{ph} - p_{el} = p_{ph} + |p_{el}|$

$|p_{el}| = \sqrt{2mE_{el}} = \sqrt{2m\left(\frac{hc}{\lambda} - \phi\right)}$ 5pts

$\frac{hc}{\lambda} - \phi = 6.2\,eV - 4.3\,eV = 1.9\,eV$

$|p_{el}| = \sqrt{2\cdot 9.11\times 10^{-31}\,kg\cdot 1.9\,eV\cdot 1.60\times 10^{-19}\,\frac{J}{eV}} = 7.44\times 10^{-25}\,\frac{kg\cdot m}{s}$

$p = 7.47\times 10^{-27}\,\frac{kg\cdot m}{s}$ 5 pts

A2. QM short

An operator Q is called anti-hermitian when it equals *minus* its hermitian conjugate:

$Q = -Q^{\dagger}$.

a. Show that the commutator of two hermitian operators is anti-hermitian.
b. Is the commutator of two *anti*-hermitian operators also anti-hermitian?
c. Show that the eigenvalues of an anti-hermitian operator are purely imaginary.

Answers

Part a.

For A and B both hermitian:

$[A,B]^{\dagger} = (AB - BA)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB = -[A,B]$

8 pts

Part b.

For R and S both anti-hermitian:

Yes it is: $[R,S]^{\dagger} = (RS - SR)^{\dagger} = S^{\dagger}R^{\dagger} - R^{\dagger}S^{\dagger} = (-S)(-R) - (-R)(-S) = SR - RS = -[R,S]$

8 pts

Part c.

$$\mathrm{G}|g\rangle = g|g\rangle \;\Rightarrow\; \langle g|\mathrm{G}|g\rangle = \langle g|g|g\rangle = g$$
$$g^* = (\langle g|\mathrm{G}|g\rangle)^* = \langle g|\mathrm{G}^{\dagger}|g\rangle = \langle g|(-\mathrm{G})|g\rangle = -\langle g|\mathrm{G}|g\rangle = -g$$
$$g^* = -g \;\Rightarrow\; g \in i\mathbb{R} = \{x\,|\,\mathrm{Re}[x] = 0\}$$

9 pts

A3. QM short

For what de Broglie wavelength is the kinetic energy of an electron equal to the energy of a photon with wavelength 1 nm?

Answer

(a) Assuming the electron is nonrelativistic, we have: Kinetic energy of electron = $K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda_{dB}^2}$

and energy of photon = $E = \frac{hc}{\lambda}$.

7 pts

Setting these energies equal, we find

$$K = E \Rightarrow \frac{h^2}{2m\lambda_{dB}^2} = \frac{hc}{\lambda} \Rightarrow \lambda_{dB}^2 = \frac{h}{2mc}\lambda = \lambda_c\lambda \Rightarrow$$

extra /2 under square root

$$\lambda_{dB} = \sqrt{\lambda_c\lambda} = \sqrt{\lambda_c\lambda} = \sqrt{2.43\times10^{-12}\times1\times10^{-9}} = 4.93\times10^{-11} = 49.3\text{ pm}$$

6 pts

Answer: 34.9pm

(b) Photon's energy is hc/λ=1240 eV=1.24 keV. And the electron energy is the same which is much smaller than the electron rest energy 511 keV, therefore the nonrelativistic expression for K is justified.

12 pts

(A4) QM

(a) $\sin kx = \frac{1}{2i}[e^{ikx} - e^{-ikx}]$ 5pts

outcomes of the measurment: $+\hbar k$ and $-\hbar k$
with probabilities 50% each 4pts
corresponding wave functions e^{ikx} and e^{-ikx}

(b) parity $= -1$ with 100% probability 8pts
wave function $\sin kx$

(c) $t > 0$ $\psi(x,t) = \sin kx \, e^{-iEt/\hbar}$
where $E = \frac{\hbar^2 k^2}{2m}$ 8pts

B1.

1)

pb #1: (Hard) QM

$$\psi(x) = A\,(x/x_0)^n\, e^{-x/x_0}\,,$$ A, n, x_0 are constants

a) use $H\psi = E\psi$ to find $V(x)$ and E.

6pts

$$H\psi = E\psi \iff -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = [E - V(x)]\,\psi(x) \quad (1)$$

- $$\frac{d}{dx}\psi = A\frac{n}{x_0}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} + A\left(\frac{x}{x_0}\right)^n\left(-\frac{1}{x_0}\right)e^{-x/x_0} \quad (2)$$
- $$\frac{d^2}{dx^2}\psi = A\frac{n(n-1)}{x_0^2}\left(\frac{x}{x_0}\right)^{n-2} e^{-x/x_0} - A\frac{n}{x_0^2}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0}$$

$$- A\frac{n}{x_0^2}\left(\frac{x}{x_0}\right)^{n-1} e^{-x/x_0} + A\frac{1}{x_0^2}\left(\frac{x}{x_0}\right)^n e^{-x/x_0}$$

$$= A\frac{n(n-1)}{x_0^2}\left(\frac{x}{x_0}\right)^{n-2} e^{-x/x_0} - 2A\frac{n}{x_0^2}\left(\frac{x}{x_0}\right)^{n-1}$$

$$+ A\frac{1}{x_0^2}\left(\frac{x}{x_0}\right)^n e^{-x/x_0}$$

$$= A\,(x/x_0)^n e^{-x/x_0}\left[\frac{1}{x_0^2} - \frac{2n}{x_0 x} + \frac{n(n-1)}{x^2}\right]$$

$$= \left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} - \frac{1}{x_0^2}\right]\psi(x) \quad (3)$$

⑥ (3) in (1) leads to:

$$E - V(x) = -\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x} + \frac{1}{x_0^2}\right] \quad (4)$$

• as $x \to \infty$, $V(x) \to 0$

$$\Rightarrow \boxed{E = -\frac{\hbar^2}{2m x_0^2}} \quad (5)$$

6 pts

(5) in (4) gives:

$$\boxed{V(x) = + \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - \frac{2n}{x_0 x}\right]} \quad (6)$$

b) the effective radial potential for H is $k\frac{e^2}{r} - \frac{\ell(\ell+1)\hbar^2}{2mr^2}$. Comparing this expression with

4 pts

$V(x)$ by (6), one sees that the $1/r^2$ term is formally identical with the $1/x^2$ term with the angular momentum ℓ taking the place of $n-1$. The $1/x$ term of $V(x)$ depends on $n = \ell+1$, while the $1/r$ (Coulomb) term in the effective potential of H is independent of the orbital angular momentum ℓ. This is the difference between the two potentials.

9 pts

B2.

problem #4:

Suppose the magnetic field B arises from a vector potential $\vec{A}$. Then, the velocity components of the particle are

$$\hat{v}_i = \frac{\hat{P}_i}{m} - \frac{q}{c}\frac{\hat{A}_i}{m} \qquad (1)$$

8 pts

Hence

$$[\hat{v}_i, \hat{v}_j] = \frac{1}{m^2}\left[\hat{P}_i - \frac{q}{c}\hat{A}_i, \hat{P}_j - \frac{q}{c}\hat{A}_j\right]$$

$$= \frac{q}{m^2 c}\left\{[\hat{P}_j, \hat{A}_i] - [\hat{P}_i, \hat{A}_j]\right\}$$

$$= \frac{i\hbar q}{m^2 c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$$

17 pts

$$= \frac{i\hbar q}{m^2 c}\sum_{k=1}^{3} \varepsilon_{ijk} B_k$$

where ε_{ijk} is the Levi-Civita density. We have used

$$\begin{cases} \vec{\nabla}\times\vec{A} = \vec{B} \\ \hat{P}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i} \end{cases}$$

Notes on B2;

1/c factor appears in Gaussian units. Most likely, they will do this problem in SI units without 1/c

The use of the Levi-Civita symbol is not required. It would suffice to get $[v_1,v_2]$ (or $[v_x,v_y]$), and then other two by cyclic permutation.

B3 Consider an electron in a uniform magnetic field pointing along the z direction. At time $t = 0$, the electron spin is along the positive y direction, with spin state vector

$$|\psi\rangle = \tfrac{1}{2}\sqrt{2}\left(|\uparrow\rangle + i|\downarrow\rangle \right) = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

a. Show that $|\psi\rangle$ is an eigenstate of S_y, and give the eigenvalue.
b. Find the spin state vector for $t > 0$.
c. Also give the expectation value of S_z for $t > 0$.

Answers

$S_y|\psi\rangle = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix} = \tfrac{1}{4}\hbar\sqrt{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix} 1 \\ i \end{pmatrix}$. The eigenvalue is $+\tfrac{1}{2}\hbar$.

8 pts

(B3) (b)

eigenstates of $H = \gamma' S_z B$ are

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalues

$\pm \gamma' \frac{\hbar}{2} B = \pm \frac{e\hbar}{2m} B = \pm \omega_L$

where ω_L is the Larmor frequency

Therefore

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t/2} \\ i e^{i\omega t/2} \end{pmatrix}$$

8 pts

(c)

$$\langle \sigma_z \rangle = \frac{1}{2} \left(e^{i\omega t/2}, -i e^{-i\omega t/2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2} \\ i e^{i\omega t/2} \end{pmatrix}$$

$$= \frac{1}{2} \left(e^{i\omega t/2}, -i e^{-i\omega t/2} \right) \begin{pmatrix} e^{-i\omega t/2} \\ -i e^{i\omega t/2} \end{pmatrix} = 0 \qquad \langle S_z \rangle = 0$$

$$\langle \sigma_x \rangle = \frac{1}{2} \left(e^{i\omega t/2}, -i e^{-i\omega t/2} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2} \\ i e^{i\omega t/2} \end{pmatrix}$$

$$= \frac{1}{2} \left(e^{i\omega t/2}, -i e^{-i\omega t/2} \right) \begin{pmatrix} i e^{i\omega t/2} \\ e^{-i\omega t/2} \end{pmatrix}$$

$$= \frac{i}{2} \left(e^{i\omega t} - e^{-i\omega t} \right) = -\sin \omega t$$

$$\langle S_x \rangle = -\frac{\hbar}{2} \sin \omega t$$

9 pts

QM (B4)

(a) $\int \psi^2(x)\,dx = C^2 \int [\varphi_0^2(x) + \varphi_1^2(x)]\,dx = 2C^2 = 1$

$$C = \frac{1}{\sqrt{2}}$$

5 pts

(b) $\langle x \rangle = \int x\,\psi^2(x) = \frac{1}{2}\int x\,[\varphi_0^2 + \varphi_1^2 + 2\varphi_0\varphi_1]\,dx$

First two integrals = 0 because integrand is an odd function

$$\langle x \rangle = \int x\,\varphi_0(x)\varphi_1(x)\,dx = \frac{\alpha}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} x\,e^{-\alpha^2x^2}\cdot 2\alpha x\,dx$$

$\alpha^2 x^2 = t$
$2\alpha^2 x\,dx = dt$

$$= \frac{2}{(2\pi)^{1/2}}\int_0^{\infty} \frac{t^{1/2}}{\alpha} e^{-t}\,dt = \frac{2}{(2\pi)^{1/2}\alpha}\,\frac{1}{2}\,\pi^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$$

alternatively, one can use the Fock representation

$$\langle 0|x|1\rangle = \frac{1}{\alpha\sqrt{2}}\langle 0|a_+ + a_-|1\rangle = \frac{1}{\alpha\sqrt{2}} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$$

$$a_\pm = \frac{1}{\sqrt{2}}\left[\alpha x \mp i\frac{p}{\hbar\alpha}\right];\quad x = \frac{a_+ + a_-}{\alpha\sqrt{2}}$$

10 pts

(c) $\psi(x,t) = \frac{1}{\sqrt{2}}[\varphi_0 e^{-iE_0t/\hbar} + \varphi_1 e^{-iE_1t/\hbar}]\qquad E_0 = \frac{\hbar\omega}{2},\ E_1 = \frac{3\hbar\omega}{2}$

$$\langle x \rangle(t) = \frac{1}{2}\int x\,[\tilde\varphi_0^*\tilde\varphi_1 + \tilde\varphi_0\tilde\varphi_1^*]\,dx = \mathrm{Re}\left[\int x\,\tilde\varphi_1^*\tilde\varphi_0\,dx\right]$$

$$\tilde\varphi_n = \varphi_n e^{-iE_nt/\hbar}$$

$$= \mathrm{Re}\left[e^{i\omega t}\int x\,\varphi_0\varphi_1\,dx\right] = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\cos\omega t$$

10 pts