A1

8 pts

(a) Conservation of momentum suggest that

$$\left[p_{photon\,i} + p_{pan\,i} = p_{photon\,f} + p_{pan} \right] \quad 2\text{ pts.}$$

Note that the copper ~~crepe~~ pan is initially at rest and the photon is absorbed after the collision, therefore,

$$\left[\frac{h}{\lambda} + 0 = 0 + p_{pan} \right] \quad 3\text{ pts.}$$

So we obtain

$$\left[p_{pan} = \frac{h}{\lambda} = \frac{6.63\ x\ 10^{-34}\ J \cdot s}{100\ x\ 10^{-9} m} = 6.63\ x\ 10^{-27} \frac{kg \cdot m}{s} \right] \quad 2\text{ pts.}$$

8 pts

b) If the copper crepe pan has a work function of 4.5 eV, what is the energy of the photoelectron emitted as a result the absorption of the photon in part (a) ? Recall that $1.6x10^{-19}$ Joules = 1eV.

(b) Using the photoelectric effect equation (Conservation of energy),

$$\left[E_{photon} = K_e + \Phi \right] \quad 3\text{ points.}$$

We can solve for the kinetic energy of the photoelectron to obtain

5 points

$$K_e = E_{photon} - \Phi = \frac{hc}{\lambda} - \Phi = \frac{1240\ eV \cdot nm}{100\ nm} - 4.5eV = 12.4\ eV - 4.5\ eV = 7.9\ eV$$

$$= 7.9\ eV\ or\ 1.26\ x\ 10^{-18} J$$

9 points

c)

Conservation of momentum require that

$$\left[p_{pan_f} = p_{photon} + p_{pan} \right] \quad 3\text{ points.}$$

$P_f = P_i$

$P_{pan} + P_{electron} = P_{electron}$ ~~photon~~ pan initial

$P_{f\,pan} = P_{pan\ initial} - P_{electron}$

Therefore,

3 pts.

$$\boxed{p_{pan_f} = \frac{h}{\lambda} + \sqrt{2m_e K_e}} = \boxed{\frac{6.63 \ x \ 10^{-34} \ J \cdot s}{100 \ x \ 10^{-9} \ m} + \sqrt{2(9.11 \ x \ 10^{-31} kg)(1.26 \ x \ 10^{-18} J)}}$$

$$\boxed{= 1.52 \ x \ 10^{-24} \frac{kg \cdot m}{s}}$$

3 points

A2

The parity operator $\hat{P}$ in 1 dimension is defined by $\hat{P}\psi(x) = \psi(-x)$

a. Is $\hat{P}$ Hermitian?
b. Find the eigenvalues of $\hat{P}$.
c. Is e^{ikx} eigenstate of P?

ANSWERS

Let $-x = u \Rightarrow -dx = du$. switch limits.

9 points a. For all $|\psi\rangle, |\varphi\rangle$, we have $\langle\varphi|\hat{P}^{\dagger}|\psi\rangle = \langle\hat{P}\varphi|\psi\rangle = \int_{-\infty}^{\infty} \varphi^*(-x)\psi(x)dx = \int_{-\infty}^{\infty} \varphi^*(u)\psi(-u)du = \langle\varphi|\hat{P}|\psi\rangle$, so $\hat{P}$ is Hermitian

8 points b. If we have $\hat{P}\psi(x) = \lambda\psi(x)$, then $\hat{P}\hat{P}\psi(x) = \psi(x) = \lambda^2\psi(x) \Rightarrow \lambda^2 = 1$, so $\lambda = \pm 1$

8 pts. c. e^{ikx} is not an eigenstate of P since $Pe^{ikx} = e^{-ikx}$

A3.

5 points a) The de Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K_e}} = \frac{6.63 \ x \ 10^{-34} J \cdot s}{\sqrt{2(9.11 \ x \ 10^{-31} \ kg)\left(50 \ eV \ x \ \frac{1.60 \ x \ 10^{-19} J}{1 \ eV}\right)}} = 1.74 \ x \ 10^{-10} \ m$$

10 points b) If the electrons are monoenergetic but <u>not</u> coherent (not single slit diffraction) and are fired through a slit that is 1 nm wide, estimate the width of the electron beam 1 meter after the electrons go through the slit?

Since the electron can go through any point within the slit along the y direction, we can then say that the uncertainty in position of finding the electron within the slit is given by

$$\boxed{\Delta y = 1 \ nm}$$

2 points.

Using Heisenberg principle

$$\Delta y \Delta P_y \geq \frac{\hbar}{2}$$

we obtain that

$$\Delta v_y \geq \frac{\hbar}{2m_e \Delta y}$$

2 points

Notice that the travel time of the electron along the x direction is given by,

$$t = \frac{d}{v_x}$$

Since we know the kinetic energy of the electron, we can rewrite the above as

$$t = \frac{d}{\sqrt{\frac{2K_e}{m_e}}} = d\sqrt{\frac{m_e}{2K_e}}$$

3 points

The width of the electron beam is then given by,

$$W = \Delta v_y t = \left(\frac{\hbar}{2m_e \Delta y}\right)\left(d\sqrt{\frac{m_e}{2K_e}}\right) = \frac{\hbar d}{\Delta y}\sqrt{\frac{1}{8m_e K_e}}$$

$$= \frac{(1.05\ x\ 10^{-34} J \cdot s)(1m)}{(1\ x\ 10^{-9}\ m)}\sqrt{\frac{1}{8(9.11\ x\ 10^{-31}\ kg)\left(50\ eV\ x\ \frac{1.60\ x\ 10^{-19} J}{1\ eV}\right)}}$$

$$= 1.38\ x\ 10^{-2} m$$

3 points

Alternate approach:

From the single slit configuration we can obtain the relation

$$2\tan\theta = \frac{\Delta p_y}{p_x}$$

Assuming that θ is small we know that $\tan\theta = \sin\theta$, therefore the above becomes,

$$2\sin\theta = \frac{\Delta p_y}{p_x}$$

Using the Heisenberg uncertainty principle we obtain

$$2\sin\theta = \frac{\hbar}{2\Delta y p_x}$$

which is equivalent to

$$2\sin\theta = \frac{\hbar}{2\Delta y\sqrt{2m_e E}}$$

Since the width of the electron beam is given by

$$W = 2d\sin\theta$$

we have

$$W = \frac{\hbar d}{2\Delta y\sqrt{(2m_e E)}}$$

$$W = \frac{(1.05\ x\ 10^{-34} J\cdot s)(1m)}{2(1\ x\ 10^{-9}\ m)}\sqrt{\frac{1}{2(9.11\ x\ 10^{-31}\ kg)\left(50\ eV\ x\ \frac{1.60\ x\ 10^{-19} J}{1\ eV}\right)}}$$

$$= 1.38\ x\ 10^{-2} m$$

The answer $2.76\ x\ 10^{-2} m$ (considering the +y and –y direction) will be given full credit too.

10 points c) Using the values above, but now assume the electron beam is coherent, with a double slit aperture, with the slits separated by 3 nm, what is the scattering angle to the second peak (with the angle defined by the deviation from unperturbed direction of propagation).

Solution

The condition for constructive interference in a double slit experiment is given by

$$\boxed{d \sin\theta = n\lambda}$$

3 points

n = 2

Solving for the scattering angle θ to the second peak ($n = 3$) we obtain,

3 points

$$\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right) = \sin^{-1}\left(\frac{3(1.74\ x\ 10^{-10}\ m)}{(3.0\ x\ 10^{-9}\ m)}\right) = 10.0°$$

4 points

B1.

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by $|\alpha\rangle = i|1\rangle + 2|2\rangle + i|3\rangle$ and $|\beta\rangle = i|1\rangle + 2|3\rangle$.

a. Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$.
b. Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$.
c. Find all nine matrix elements of the operator $\hat{A} = |\alpha\rangle\langle\beta|$ and construct its matrix **A** in the basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Is **A** hermitian?

ANSWERS

8 points a. $\langle\alpha| = |\alpha\rangle^{\dagger} = \left(i|1\rangle + 2|2\rangle + i|3\rangle\right)^{\dagger} = \left(i|1\rangle\right)^{\dagger} + \left(2|2\rangle\right)^{\dagger} + \left(i|3\rangle\right)^{\dagger} = -i\langle 1| + 2\langle 2| - i\langle 3|.$ 4 pts

Similarly, $\langle\beta| = -i\langle 1| + 2\langle 3|$.] 4pts.

b. $\langle\alpha|\beta\rangle = (-i\langle 1| + 2\langle 2| - i\langle 3|)(i|1\rangle + 2|3\rangle) = (-i)i\langle 1|1\rangle + (-i)2\langle 3|3\rangle = 1 - 2i$
$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^* = 1 + 2i$] 7 points.

10 points

c. $\hat{A} = |\alpha\rangle\langle\beta| = (i|1\rangle + 2|2\rangle + i|3\rangle)(-i\langle 1| + 2\langle 3|) =$
$= i(-i)|1\rangle\langle 1| + 2(-i)|2\rangle\langle 1| + i(-i)|3\rangle\langle 1| + 2i|1\rangle\langle 3| + 4|2\rangle\langle 3| + 2i|3\rangle\langle 3| =$] 4 points. (remember orthonormal).

[$= |1\rangle\langle 1| - 2i|2\rangle\langle 1| + |3\rangle\langle 1| + 2i|1\rangle\langle 3| + 4|2\rangle\langle 3| + 2i|3\rangle\langle 3|$] 2 pts.

Now with $A_{ij} = |i\rangle\langle j|$ we have $\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2i \\ -2i & 0 & 4 \\ 1 & 0 & 2i \end{pmatrix}$

$A_{ji} = \begin{bmatrix} 1 & -2i & 1 \\ 0 & 0 & 0 \\ 2i & 4 & 2i \end{bmatrix}$

not $\neq [A_{ij}]^*$.

2 pts

We see that $A_{ji} \neq A_{ij}^*$, so the matrix (or operator) is not Hermitian. → 2 points.

B2. Find the eigenvalues of the component of the electron spin in the direction of an arbitrary unit vector $\hat{\boldsymbol{n}}$

ANSWERS

5 points [We must solve

$\hat{\boldsymbol{n}}\cdot\boldsymbol{S}|\lambda\rangle = \frac{\hbar}{2}\lambda|\lambda\rangle$, so we write the eigenvalues as $\lambda\frac{\hbar}{2}$.]

3 points [In spherical coordinates, the unit vector in the direction (θ,ϕ) is given by $\hat{\boldsymbol{n}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$, so we have]

2 pts. [$\hat{\boldsymbol{n}}\cdot\boldsymbol{S} = S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta =$

3 pts [$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta =$

(Pauli spin matrices are given in formula sheet)

5 pts

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} =$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

7 pts [Diagonalization:

$$-\frac{\hbar^2}{4}(\cos\theta - \lambda)(\cos\theta + \lambda) - \frac{\hbar^2}{4}\sin^2\theta = 0 \quad \Rightarrow$$

$$\cos^2\theta + \lambda\cos\theta - \lambda\cos\theta - \lambda^2 + \sin^2\theta = 0 \quad \Rightarrow$$

$$1 - \lambda^2 = 0 \quad \Rightarrow \quad \lambda = \pm 1$$]

$\Rightarrow$ eigen values $\pm \hbar/2$.

So the eigenvalues are $\pm\dfrac{\hbar}{2}$

====================================

B3. An electron in a hydrogen atom is in the stationary state

$$\psi_{2,1,-1}(r,\theta,\phi) = Nre^{-r/2a_0}Y_{1,-1}(\theta,\phi)$$

(a) Find the normalization constant *N*. Check that it has the correct unit.
(b) What is the probability per unit volume of finding the electron at $r = a_0$, $\theta = 45°$, and $\phi = 60°$?
(c) What is the probability per unit radial interval (*dr*) of finding the electron at $r = 2a_0$?
(d) If L^2 is measured, what outcomes can be found and with what probabiliies?
(e) If, instead, L_z is measured, what outcomes can be foundand with what probabilities?

Cheat sheet: $\int u^4 e^{-u} du = -e^{-u}(u^4 + 4u^3 + 12u^2 + 24u + 24)$, plus a list of low-order spherical harmonics.

<u>**ANSWERS**</u>

(a) $\int |\psi|^2 \, dV = \int |\psi|^2 \, r^2 dr d\Omega = \int N^2 r^2 e^{-\frac{r}{a_0}} r^2 dr \times \int |Y|^2 d\Omega = N^2 \int r^4 e^{-\frac{r}{a_0}} dr =$

$$N^2 \left[-a_0 e^{-\frac{r}{a_0}} (24\, a_0^4 + 24 a_0^3 r + 12\, a_0^2 r^2 + 4\, a_0 r^3 + r^4) \right]_{r=0}^{\infty} =$$

$$N^2 \left[a\, e^{-\frac{r}{a}} (24\, a^4 + 24 a^3 r + 12\, a^2 r^2 + 4\, a\, r^3 + r^4) \right]_{\infty}^{r=0} = 24\, a^5 N^2 = 1$$

$$N = \frac{1}{\sqrt{24\, a_0^5}} = \frac{\sqrt{6}}{12 a^{5/2}}$$ Why bother ?

So $\psi(r,\theta,\phi) = \frac{\sqrt{6}}{12 a_0^{5/2}} r e^{-\frac{r}{2a_0}} Y_{1,-1}(\theta,\phi)$. Check: The unit of ψ becomes 1/m$^{3/2}$, as it should.

(b) This probability density is $|\psi(r,\theta,\phi)|^2 = \frac{1}{24 a^5} r^2 e^{-\frac{r}{a_0}} \left| Y_{1,-1}(\theta,\phi) \right|^2 =$

$= \frac{1}{24a_0^5} r^2 e^{-\frac{r}{a_0}} \left| -\sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta \right|^2 = \frac{1}{24a_0^5} a_0^2 e^{-\frac{a_0}{a_0}} \frac{3}{8\pi} \sin^2(45°) = \frac{1}{24a_0^3} e^{-1} \frac{3}{8\pi} \frac{1}{2} = \frac{1}{a_0^3} \frac{3}{24*8\pi} \frac{1}{2e} =$
$\frac{1}{a_0^3} \frac{1}{128\pi e} = \frac{9.15\times 10^{-4}}{a_0^3} = \mathbf{6.18 \times 10^{27}\ m^{-3}}$, quite a large value. Should we ask HOW this >> 1 ?

(c) $dP = R^2(r) r^2 dr \times \int |Y|^2 d\Omega = N^2 r^2 e^{-\frac{r}{a_0}} r^2 dr \times 1 = N^2 r^4 e^{-\frac{r}{a_0}} dr = N^2 (2a_0)^4 e^{-\frac{2a_0}{a_0}} dr =$
$\frac{2}{3e^2 a_0} dr$

so the answer is $\frac{2}{3e^2 a_0} = \mathbf{1.71 \times 10^9\ m^{-1}}$.

(d) The only angular momentum in this state is $l = 1$, so when L^2 is measured, we find $l(l+1)\hbar^2 = 2\hbar^2$ with 100% probability.

(e) Similarly, we only have $m_l = -1$, so we measure $L_z = -\hbar$ with certainty.

QM

A4 (a) $\langle A^2 \rangle = \langle \psi^* | A^2 \psi \rangle = \langle A\psi | A\psi \rangle \geq 0$

(squared norm)

if $A^\dagger = -A$ we have

$\langle A^2 \rangle = -\langle A\psi | A\psi \rangle \leq 0$

(b) $U = e^{iA} = \sum_n \frac{(iA)^n}{n!}$

$(e^{iA})^\dagger = \sum_n \frac{(-iA^\dagger)^n}{n!} = \sum_n \frac{(-iA)^n}{n!} = e^{-iA} = U^{-1}$

$e^{iA} e^{-iA} = 1$

QM

B4. (a) $S_x \chi = \cancel{\hbar} \lambda \chi$

$$\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \lambda\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$$

$$\frac{\hbar}{2}\alpha_2 = \lambda\alpha_1$$
$$\frac{\hbar}{2}\alpha_1 = \lambda\alpha_2 \rightarrow \lambda = \pm\frac{\hbar}{2}$$

$$\alpha_2 = \pm\alpha_1$$

$$\chi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

6 pts.

(b)
2 [$\langle S_z\rangle = \frac{\hbar}{2}\langle\sigma_z\rangle = \frac{\hbar}{2}\cdot\frac{1}{2}(1\ 1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{4}(1\ 1)\begin{pmatrix}1\\-1\end{pmatrix} = 0$

2 [$\langle\sigma_z^2\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 1 \qquad \langle S_z^2\rangle = \frac{\hbar^2}{4}$

2 [$\langle\sigma_y\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 0$

2 [$\langle\sigma_y^2\rangle = 1 \qquad \langle S_y^2\rangle = \frac{\hbar^2}{4}$

4 pts [(c) $\langle\vec{S}^2\rangle = \langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{3}{4}\hbar^2$

(d)

3 points [~~$H = \frac{e}{m}\frac{\hbar}{2}B = \frac{e\hbar}{2m}B$~~

$H = \frac{e}{m}\vec{S}\cdot\vec{B} = -\frac{e}{m}S_x B$

yes, because H commutes with S_x]

4 pts. [$E = \mp\frac{e}{m}\frac{\hbar}{2}B = \mp\frac{e\hbar}{2m}B$

A1 The electrostatic field produced by a charge density is given by

$$\mathbf{E} = E_0 \frac{y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r}.$$

Find the total charge Q contained in a sphere of radius $\boxed{a}$ R_o centered at the origin.

Solution:

5pts

According to the Gauss's law, the total charge is

$$Q = \varepsilon_0 \int_S (\mathbf{E}\cdot\hat{\mathbf{r}})da,$$

where the integration i~~f~~ s performed over the sphere of radius ~~a~~ R_o. In spherical coordinates, we have

6pts

$\hat{\mathbf{r}} = \cancel{\sin\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}}$ | $\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

Commented [IF1]: ~~More conventionally, cos\phi goes with x, and sin\phi with y, but this does not change the result~~

and on the sphere

6pts

$$\mathbf{E}\cdot\hat{\mathbf{r}} = E_0\left[\sin^2\theta\cos\phi\sin\phi + \sin^2\theta\sin\phi\cos\phi + \cos^2\theta\right] = E_0\left[\sin^2\theta\sin 2\phi + \cos^2\theta\right].$$

R_o R_o R_o R_o

Commented [IF2]: ~~Sin(2\pi) should be divided by 2, but this does not change the final result~~ NOT.

The integration of $\sin 2\phi$ over ϕ gives zero, and therefore we obtain: 3pts

3pts

from $d\Phi$

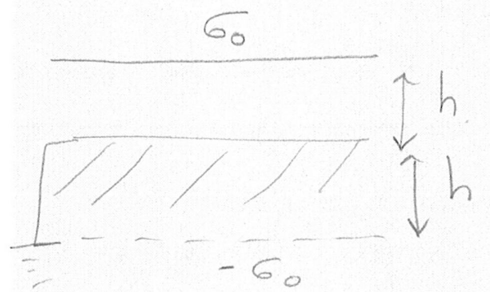
$$Q = 2\pi\varepsilon_0 a^2 E_0 \int_0^\pi \cos^2\theta\sin\theta d\theta = \boxed{\frac{4}{3}\pi\varepsilon_0 a^2 E_0}.$$

2 points

$\therefore Q = \epsilon_0 \oint E\cdot\hat{r}\,da$ $da = R_o\,d\theta\,R_o\sin\theta\,d\Phi$

$\therefore Q = \epsilon_0 \int_0^{2\pi}\int_0^{\pi} E_0 [\quad] R_o^2 \sin\theta\,d\theta\,d\Phi$

Should we include this integral in formula sheet?



A2. An infinite plane of a uniform charge σ_0 per unit area is placed at distance $z = h$ above the surface of a half-space grounded metal.

15 pts (1) Find the 1st potential and the electric field in all space.

10 pts (2) Find the induced surface charge on ~~a~~ the grounded metal surface.

Solution:

3 (1) The solution can be obtained using a method of images. An image charge plane is located at $z = -h$ below the surface of a metal and has charge $-\sigma_0$ per unit area. The electric ~~file~~ field produced

3pts by the two planes of charge is uniform between the planes and is equal to $\mathbf{E} = -\frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{z}}$. Above the

2pts plane of charge, $z > h$, the electric field is zero. Since $\mathbf{E} = -\nabla\Phi$, we find that between the planes the potential is $\Phi(z) = \frac{\sigma_0}{\varepsilon_0}z + C$, where constant C must be equal to zero in order to have zero potential on the grounded metal surface. Above the plane of charge the potential is constant and is equal to $\Phi(z) = \frac{\sigma_0}{\varepsilon_0}h$. Thus,

$$\Phi(z) = 0; \quad \mathbf{E} = 0; \quad z < 0$$

$$\boxed{\Phi(z) = \frac{\sigma_0}{\varepsilon_0}z; \quad \mathbf{E} = -\frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{z}}; \quad 0 < z < h,} \tag{1}$$

$$\Phi(z) = \frac{\sigma_0}{\varepsilon_0}h; \quad \mathbf{E} = 0; \quad z > h$$

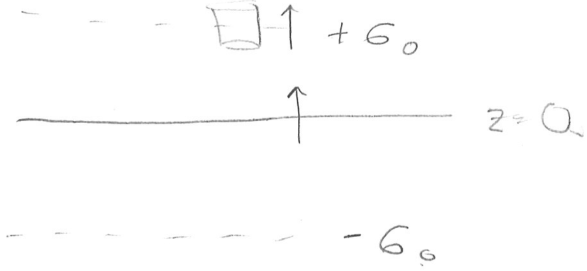
at $z = 0$ and $z = h$ the potential is continuous, while the electric field is discontinuous.

(2) The induced surface charge density σ on ~~a~~ the metal surface is given by

$$\boxed{\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}},} \quad 5 \text{ pts} \tag{2}$$

$\mathbf{E}_{above}$ is the electric field above the metal surface, $\mathbf{E}_{below}$ is the electric field below the metal surface, and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ is the normal to the surface. From eqs.(1) and (2) we find that $\sigma = -\sigma_0$.

Use method of images



1st find potential in region between $z=0$ to $z=h$.

$$V = \frac{\sigma_0}{\epsilon_0} z \text{ for } 0 < z < h$$

$$V = 0 \text{ for } z < 0 \quad \text{(grounded)}$$

$V =$

$$\bar{E} \text{ for } 0<z<h = -\nabla V = -\frac{d}{dz}\frac{\sigma_0}{\epsilon_0} z \hat{z} = -\frac{\sigma_0}{\epsilon_0}\hat{z} \quad \text{(points down)}$$

$\bar{E}$ for $h > z$ $\quad \bar{E}_{out} - \bar{E}_{in} = \frac{\sigma}{\epsilon_0}$

$$\therefore E_{out} = \frac{\sigma_0}{\epsilon_0} - \frac{\sigma_0}{\epsilon_0} = 0.$$

At $z=0$, $\quad \bar{E}_{out} - E_{in} = \frac{\sigma_{induced}}{\epsilon_0}$.

$$\therefore -\frac{\sigma}{\epsilon_0} - 0 = \frac{\sigma_{induced}}{\epsilon_0} \quad \therefore \sigma_{induce} = -\sigma_0.$$

EM

A3.

5 pts   5 pts

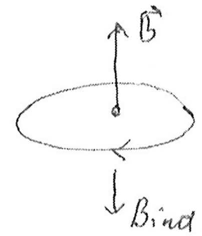
(a) $$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}} \boxed{= -B\frac{d}{dt}\left[\pi(r_0+vt)^2\right] =}$$

$$-2\pi B v(r_0+vt) = -2\pi B v r$$

5 pts

(b)

10 points



(Lenz law)

If $\bar{B}$ points upwards current is CW as viewed from top

EM

A4

$$B_x = a(x-y)\,,\quad B_y = -a(y+z)/2\,,\quad B_z = -a(x+z)/2$$

8 pts

(a) check $\vec\nabla\cdot\vec B = 0$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = a - \frac{a}{2} - \frac{a}{2} = 0$$

yes

9 pts

(b) $\vec\nabla\times\vec B = \mu_0 \vec J$

$$\vec\nabla\times\vec B = \begin{vmatrix} \hat x & \hat y & \hat z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a(x+y) & -\frac{a}{2}(y+z) & -\frac{a}{2}(x+z) \end{vmatrix} = \hat x \frac{a}{2} + \hat y \frac{a}{2} - \hat z\, a$$

$$\vec J = \frac{a}{\mu_0}\left(\frac{1}{2}\hat x + \frac{1}{2}\hat y - \hat z\right)$$

8 pts

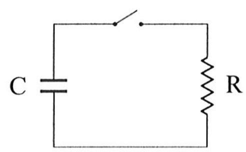
(c) $\vec\nabla\cdot\vec J = \frac{a}{\mu_0}\left(\frac{1}{2} + \frac{1}{2} - 1\right) = 0$

and from continuity eq. $\frac{\partial \rho}{\partial t} = 0$

given ~~Should we give the continuity equation on formula sheet?~~

$\cancel{\vec\nabla\cdot\vec J = -\frac{\partial \rho}{\partial t}}$

B1: A simple circuit consists of a capacitor C carrying charge Q and a resistor R as shown. At time $t = 0$, the switch is closed and the capacitor is discharging. Find current $I(t)$ as a function of time t and show that the electric energy originally stored on the capacitor is fully dissipated on resistor R at $t \to \infty$.



Solution:

1. When the switch is closed, current I flows through the circuit. According to Kirchhoff's loop rule

5 pts

$$V_R + V_C = 0,$$

where V_R is a voltage drop on the resistor $V_R = IR$ and V_C is a voltage on the capacitor. The latter is given by $V_C = q/C$, where q is a charge on the capacitor (dependent on time). Due to the charge conservation the current I in the circuit is determined by the same charge, so that $I = dq/dt$. We therefore obtain:

5 pts

$$R\frac{dq}{dt} + \frac{q}{C} = 0,$$

resulting in $q(t) = Qe^{-t/RC}$, where Q is the charge on the capacitor at $t = 0$. The current I is then given by

5 pts

$$I(t) = \frac{dq}{dt} = -\frac{Q}{RC}e^{-\frac{t}{RC}}.$$

2. The initial energy stored on the capacitor is $\boxed{W_C = \frac{1}{2}\frac{Q^2}{C}}$ 3 pts During the discharge of the capacitor the energy per unit time dissipated on the resistor is I^2R. Integrating over time, we obtain for the total energy dissipated:

7 pts

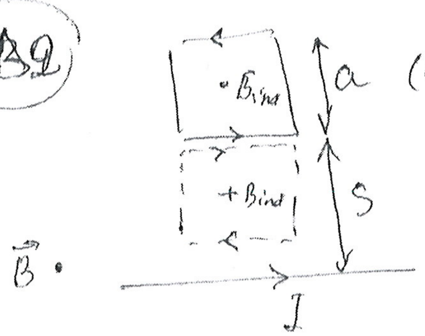
$$W_R = \int_0^\infty I^2(t)Rdt = \frac{Q^2}{RC^2}\int_0^\infty e^{-\frac{2t}{RC}}dt = \frac{Q^2}{RC^2}\frac{RC}{2} = \frac{1}{2}\frac{Q^2}{C}.$$

We see therefore that $W_C = W_R$, as required.

EM

10 pts

B2



(a) Direction of the current follows from the Lenz law: initially Φ is decreasing, therefore $\vec{B}_{ind}$ has the same direction as $\vec{B}$.
At the end Φ is increasing in abs value, therefore $\vec{B}_{ind}$ is opposite to $\vec{B}$

(b) For the initial position

$$B = \frac{\mu_0 I}{2\pi s'}$$

$$\boxed{\Phi_1 = \int_s^{s+a} B a\, ds' = \frac{\mu_0 I a}{2\pi} \int_s^{s+a} \frac{ds'}{s'} = \frac{\mu_0 I a}{2\pi} \ln \frac{s+a}{s}}$$

5 pts.

For the final position

5 pts

$$\boxed{\text{Flux } \Phi_2 = \frac{\mu_0 I a}{2\pi} \int_{s-a}^{s} \frac{ds'}{s'} = -\frac{\mu_0 I a}{2\pi} \ln \frac{s}{s-a}}$$

(sign – because the orientation changes)

5pts

$$\Delta \Phi = \Phi_2 - \Phi_1 = -\frac{\mu_0 I a}{2\pi} \ln \frac{s+a}{s-a}$$

$$Q = -\frac{\Delta \Phi}{R} = \frac{\mu_0 I a}{2\pi R} \ln \frac{s+a}{s-a}$$

(sign of charge is NOT fussed)

positive sign means that net charge is flowing in the direction shown in the upper square

B3. A dielectric sphere of radius R with a dielectric constant ε has a free charge Q uniformly distributed over the volume. The sphere is surrounded by empty space.

1. Find electric field **E** and electrical displacement **D** inside and outside the sphere.
2. Find polarization **P** and the volume and surface polarization charge densities.
3. Show that the total polarization charge is zero.

Solution:

1. The electric displacement **D** can be found from the Gauss law.

$$\oint_S \mathbf{D}\cdot\mathbf{n}da = q\,.$$

5 pts

By symmetry **D** is pointing along the $\hat{\mathbf{r}}$ direction and spherically symmetric. Using the sphere of radius $r < R$ we therefore find

$$D4\pi r^2 = \frac{3Q}{4\pi R^3}\frac{4}{3}\pi r^3$$

and thus

$$\mathbf{D} = \frac{Qr}{4\pi R^3}\hat{\mathbf{r}}\,.$$

3 pts

Outside the sphere, $r > R$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}\,.$$

2 pts

The electric field **E** is

$$\mathbf{E} = \begin{cases} \dfrac{\mathbf{D}}{\varepsilon} = \dfrac{Qr}{4\pi\varepsilon R^3}\hat{\mathbf{r}}, & r < R \\ \dfrac{\mathbf{D}}{\varepsilon_0} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > R \end{cases}.$$

2. The electric field **E**, the polarization **P** and the electric displacement **D** are related as follows

$$\boxed{\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}\,.}$$

? Should this be in the formula sheet?

4 pts

Therefore inside the sphere

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E} = -(\varepsilon - \varepsilon_0)\frac{Qr}{4\pi\varepsilon R^3}\hat{\mathbf{r}}\,.$$

3 pts

The volume polarization charge is given by

$$\rho_P = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = \frac{\varepsilon - \varepsilon_0}{\varepsilon}\frac{3Q}{4\pi R^3}\,.$$

3pts

The surface polarization charge is given by

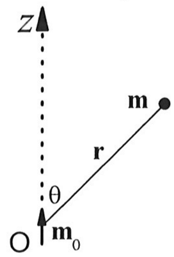
$$\sigma_P = \mathbf{P}\cdot\hat{\mathbf{r}}\big|_{r=R} = -\frac{\varepsilon - \varepsilon_0}{\varepsilon}\frac{Q}{4\pi R^2}\,.$$

5 pts.

3. The total polarization charge is therefore

$$\rho_P\frac{4\pi}{3}R^3 + \sigma_P 4\pi R^2 = 0\,.$$

B4. A magnetic moment with fixed magnitude and direction, $\mathbf{m}_0 = m_0\hat{\mathbf{z}}$, is held at the origin of coordinates. Another magnetic moment $\mathbf{m}$ is held fixed at an arbitrary point $\mathbf{r}$, but its orientation is allowed to change freely. See the figure below.
(a) Find the equilibrium orientation of the moment $\mathbf{m}$ (given by a unit vector $\hat{\mathbf{m}}$) that corresponds to the minimum of magnetostatic energy.
(b) Draw an arrow indicating the equilibrium orientation of the magnetic moment $\mathbf{m}$ on the figure below, assuming that the angle between $\mathbf{r}$ and $\mathbf{m}_0$ is $\theta = 45°$.



Solution:
(a) The interaction energy between these two magnetic moments is given by

$$U = -\mathbf{m}\cdot\mathbf{B}, \tag{1}$$

where $\mathbf{B}$ is the magnetic field generated by the moment $\mathbf{m}_0$ at the point $\mathbf{r}$, which is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3}. \tag{2}$$

From (1) we see that the minimum energy will occur when $\mathbf{m} \parallel \mathbf{B}$, so we simply need to find the orientation of the magnetic field $\hat{\mathbf{B}}$. We can rewrite (2) as

$$\mathbf{B} = \frac{\mu_0 m_0}{4\pi r^3}\left[3(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{z}}\right] = \frac{\mu_0 m_0}{4\pi r^3}\left[3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}\right]. \tag{3}$$

The modulus of this field is

$$|\mathbf{B}| = \frac{\mu_0 m_0}{4\pi r^3}\left[(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}})\cdot(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}})\right]^{1/2} = \frac{\mu_0 m_0}{4\pi r^3}\left(9\cos^2\theta + 1 - 6\cos\theta\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}\right)^{1/2}$$
$$= \frac{\mu_0 m_0}{4\pi r^3}\sqrt{3\cos^2\theta + 1}. \tag{4}$$

Therefore the equilibrium orientation of the magnetic moment $\mathbf{m}$ is

$$\hat{\mathbf{m}} = \hat{\mathbf{B}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{\sqrt{3\cos^2\theta + 1}} = \frac{3z\hat{\mathbf{r}} - r\hat{\mathbf{z}}}{\sqrt{3z^2 + r^2}}. \tag{5}$$

(b) From (5) it is clear that the magnetic moment lies in the plane formed by the vectors $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$, which without loss of generality we can take to be the xz plane. Plugging $\theta = 45°$ into (5) we find.

$$\hat{\mathbf{m}} \propto 3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}} = 3\cos\theta\left(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}\right) - \hat{\mathbf{z}} =$$
$$= \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\hat{\mathbf{x}} + \frac{\sqrt{2}}{2}\hat{\mathbf{z}}\right) - \hat{\mathbf{z}} = \frac{3}{2}\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{z}} \tag{6}$$

which should look like in the figure below.