Mechanics

A1

$\vec{L} = m\vec{r}\times\vec{v}$ 5

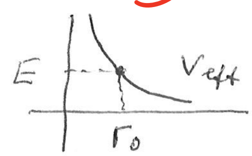
$\dot{\vec{L}} = m\vec{r}\times\dot{\vec{v}} + m\vec{v}\times\vec{v} = m\vec{r}\times\dot{\vec{v}} = \cancel{m}\vec{r}\times(\vec{f}_1+\vec{f}_2)$ 10

$= -\lambda \cancel{m}\, \vec{r}\times\vec{v} = -\lambda \frac{\vec{L}}{m}$

$\frac{d\vec{L}}{dt} = -\frac{\lambda}{m}\vec{L} \rightarrow \vec{L} = \vec{J}_0 e^{-\frac{\lambda}{m}t}$ 10

A2

The closest appoach corresponds to the turning point for the radial motion 5



Solve
$$E = \frac{L^2}{2mr^2} + \frac{k}{r} \quad \text{for } r$$

where $E = \frac{mv_0^2}{2}$, $L = mvb$ 10

$$Er^2 - kr - \frac{L^2}{2m} = 0$$

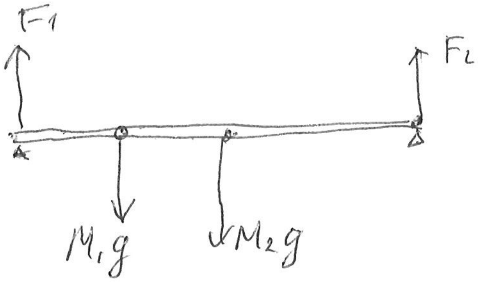
$$r_0 = \frac{k + \sqrt{k^2 + \frac{2EL^2}{m}}}{2E} = \frac{k}{2E} + \sqrt{\left(\frac{k}{2E}\right)^2 + \frac{L^2}{2Em}}$$

$$= \frac{k}{mv_0^2} + \sqrt{\left(\frac{k}{mv_0^2}\right)^2 + b^2}$$ 10

negative zero has no physical meaning

Mechanics

A3



$$\sum F = 0 \longrightarrow F_1 + F_2 - M_1 g - M_2 g = 0$$

$$\sum \tau = 0 \longrightarrow F_1 \frac{l}{2} - M_1 g \frac{l}{4} - F_2 \frac{l}{2} = 0$$

10

$$F_1 + F_2 = (M_1 + M_2) g$$

$$F_1 - F_2 = \frac{1}{2} M_1 g$$

Solve for F_1, F_2

$$F_1 = \frac{\left(\frac{3}{2} M_1 + M_2\right) g}{2}$$

10

$$F_2 = \frac{\left(\frac{1}{2} M_1 + M_2\right) g}{2}$$

$$\frac{F_1}{F_2} = \frac{\frac{3}{2} M_1 + M_2}{\frac{1}{2} M_1 + M_2}$$

5

Mechanics (A4)

(a) Conservation of angular momentum

5 $m_{child}\, v R = (m_{child} R^2 + I)\,\omega$ where $R = \frac{2.4}{2} = 1.2\,m$

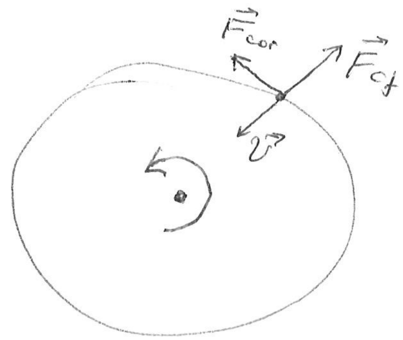
5 $$\omega = \frac{vR}{R^2 + \frac{I}{m_{child}}} = \frac{1.5\,\frac{m}{s} \cdot 1.2\,m}{(1.2)^2 m^2 + \frac{2100}{38}\,m^2} = 0.0317\,\frac{rad}{s}$$

(b) 1. Centrifugal force away from the center

7 $F_{cf} = m\omega^2 R = 38 \cdot (0.0317)^2 \cdot 1.2 = 0.046\,N$

2. Coriolis force $\vec{F}_{cor} = -2m\,\vec{\omega} \times \vec{v}$

8 magnitude $F_{cor} = 2 \cdot 38 \cdot 0.0317 \cdot 0.5 = 1.2\,N$



Mechanics

B2 [redacted] (a) Before the thrust

$$\frac{mV_1^2}{R} = \frac{GmM}{R^2} \rightarrow v_1^2 = \frac{GM}{R} \qquad (R = r_1)$$

$$\frac{T_1}{m} = \frac{v_1^2}{2} = \frac{GM}{2R} \qquad \frac{L_1}{m} = v_1 r_1 = \sqrt{GMr_1}$$

5

$$\frac{E_1}{m} = \frac{GM}{2r_1} - \frac{GM}{r_1} = -\frac{GM}{2r_1}$$

after the thrust

$$l_2 \equiv \frac{L_2}{m} = \frac{SL_1}{m} = S\sqrt{GMr_1} \qquad \frac{T_2}{m} = S^2\frac{T_1}{m} = S^2\frac{GM}{2r_1}$$

5

$$e_2 \equiv \frac{E_2}{m} = \frac{S^2GM}{2r_1} - \frac{GM}{r_1} = -\frac{GM}{r_1}\left(1 - \frac{S^2}{2}\right)$$

(b) For elliptic orbit $E_2 < 0 \rightarrow S < \sqrt{2}$ 5

Reduce score for $S \leq \sqrt{2}$

(c) apogee corresponds to the right turning point for the radial motion: $E_2 = V_{eff}(r_2)$

$$e_2 \equiv \frac{E_2}{m} = \frac{l_2^2}{2r^2} - \frac{GM}{r} \qquad \text{where } l_2 = \frac{L_2}{m}$$

$$|e_2| r^2 - GMr + \frac{l_2^2}{2} = 0$$

$$r_2 = \frac{GM + \sqrt{(GM)^2 - 2|e_2| l_2^2}}{2|e_2|} = \frac{GM + \sqrt{(GM)^2 - \frac{GM}{r_1}(2-S^2)S^2GMr_1}}{\frac{GM}{r_1}(2-S^2)}$$

$$= \frac{1 + \sqrt{(S^2-1)^2}}{2-S^2} r_1 = \frac{S^2}{2-S^2} r_1$$

10

Solution through eccentricity

$$\varepsilon = \sqrt{1 - \frac{2|e_2| l_2^2}{(GM)^2}} = S^2 - 1 \qquad \frac{r_2}{r_1} = \frac{1+\varepsilon}{1-\varepsilon} = \frac{S^2}{2-S^2}$$

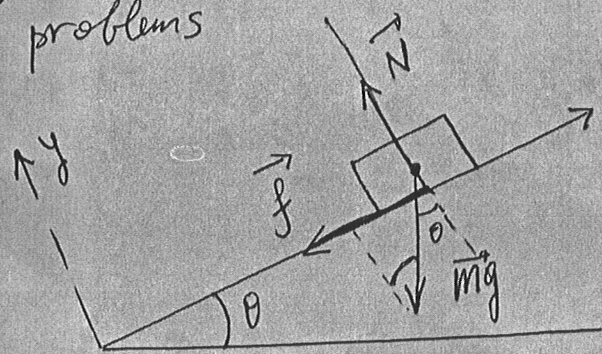
[redacted]

B3

Preliminary Exam Key, August 2023

Potential problems

5



problem #1:

$\theta = 30^\circ$

$\dot{x}_0 = +\,2\,m/s$

$\mu = \frac{1}{5\sqrt{3}}$

$f = \mu N = \mu\, mg\cos\theta$

How far is the brick from its original position? at $t = 0.5\,s$

- Newton 2nd law: $m\ddot{x} = -mg\sin\theta - \mu\, mg\cos\theta$

$$\Rightarrow \quad \ddot{x} = -g(\sin\theta + \mu\cos\theta) = -g\left(\frac{1}{2} + \frac{1}{5\sqrt{3}}\,\frac{\sqrt{3}}{2}\right)$$

$$\ddot{x} = -\frac{3}{5}g = -6\,m/s^2 \qquad 2$$

- time of upward motion of the brick: $\dot{x}(t_1) = 0$ $\quad \dot{x} = \dot{x}_0 + \ddot{x}\,t$

$$3 \quad t_1 = \frac{\dot{x}_0}{-\ddot{x}} = \frac{2\,m/s}{6\,m/s^2} = \frac{1}{3}\,s \longrightarrow t_2 = 0.5\,s$$

- displacement of the brick at $0.25\,s$ is then:

$$x_1 = \underset{\overset{\shortparallel}{0}}{x_0} + \dot{x}_0 t_1 + \frac{1}{2}\ddot{x}\,t_1^2 = 2\,m/s \times \frac{1}{3}\,s - \frac{6\,m/s^2}{2}\left(\frac{1}{3}\,s\right)^2$$

$$= \left(\frac{2}{3} - \frac{3}{9}\right)m$$

5

$$x_1 = \frac{1}{3}\,m$$

② • For $t > t_1$, $\dot{x} < 0$ and the equation of motion is:

$\searrow \vec{f}$ is now up!

$$m\ddot{x} = -mg\sin\theta + \mu mg\cos\theta$$

4

$$\ddot{x} = -g(\sin\theta - \mu\cos\theta) = -g\left(\frac{1}{2} - \frac{1}{10}\right)$$

$$\ddot{x} = -\frac{4}{10}g = -4\,m/s^2$$

• displacement during the time interval $t_1 = 1/3\,s$ to $t_2 = \frac{1}{2}\,s$ is:

$$\Delta x = \frac{1}{2}\ddot{x}\,\Delta t^2 = -\frac{1}{2}\left(4\,m/s^2\right)\left(\frac{1}{2}-\frac{1}{3}\right)^2 s^2 - 2\times\left(\frac{1}{6}\right)^2 m$$

4

$$\Delta x = -\frac{1}{18}\,m$$

• Conclusion: the displacement of the brick at $t = 0.5\,s$ is:

$$S = x_1 + \Delta x = \frac{1}{3}m - \frac{1}{18}m = \frac{5}{18}m$$

2

$$\boxed{S = \frac{5}{18}m}$$

B4 [redacted] (a) $$m\dot{\vec{v}} = -ge\,\frac{\vec{v}\times\vec{r}}{r^3}$$

10 $$m\vec{v}\cdot\dot{\vec{v}} = -ge\,\frac{(\vec{v}\times\vec{r})\cdot\vec{v}}{r^3} = 0 \quad \text{Since } \vec{v}\times\vec{r} \perp \vec{v}$$

$$\frac{d}{dt}\left(\frac{m\vec{v}^2}{2}\right) = 0 \rightarrow \frac{mv^2}{2} = \text{const}$$

(b) $$\cancel{\vec{J} = m\vec{v}\times\vec{r} + eg\frac{\vec{r}}{r}} \quad \vec{J} = m\vec{r}\times\vec{v} + eg\frac{\vec{r}}{r}$$

$$\frac{d\vec{J}}{dt} = m\vec{v}\times\vec{v} + m\vec{r}\times\dot{\vec{v}} + \frac{eg}{r^2}(\dot{\vec{r}}r - \dot{r}\vec{r})$$

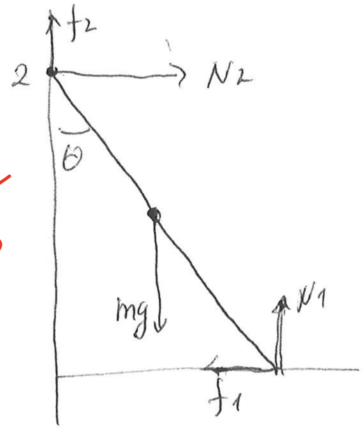
15 $$= -ge\,\frac{\vec{r}\times(\vec{v}\times\vec{r})}{r^3} + \frac{eg}{r^2}(\dot{\vec{r}}r - \dot{r}\vec{r})$$

$$\vec{r}\times(\dot{\vec{r}}\times\vec{r}) = \dot{\vec{r}}r^2 - \vec{r}(\vec{r}\cdot\dot{\vec{r}})\,;\quad \vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{1}{2}\frac{d}{dt}r^2 = \dot{r}r$$

$$\rightarrow \frac{d\vec{J}}{dt} = -ge\left(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2}\right) + \frac{eg}{r^2}(\dot{\vec{r}}r - \dot{r}\vec{r}) = 0$$

CM [redacted] B1

5



$f_1 = \mu N_1$, $f_2 = \mu N_2$

Equation for forces

$N_2 - f_1 = 0$

$N_2 - \mu N_1 = 0$

$\mu N_2 + N_1 - mg = 0$

10

$\longrightarrow \mu^2 N_1 + N_1 = mg \quad N_1 = \frac{mg}{1+\mu^2}, \quad N_2 = \frac{\mu mg}{1+\mu^2}$

Equation for torques. Choose them about point 2:

$$-f_1 l\cos\theta + N_1 l\sin\theta - mg\frac{l}{2}\sin\theta = 0$$

$$\tan\theta = \frac{\cancel{N_1}\, f_1 l}{N_1 l - mg\frac{l}{2}} = \frac{\mu N_1}{N_1 - \frac{mg}{2}} = \frac{\frac{mg\mu}{1+\mu^2}}{\frac{mg}{1+\mu^2} - \frac{mg}{2}}$$

10

$$\tan\theta = \frac{\mu}{1 - \frac{1}{2}(1+\mu^2)} = \frac{2\mu}{1-\mu^2} = \frac{2\cdot 0.4}{1-0.4^2} = 0.95$$

[redacted]

Thermo

A2

6 points Heat absorbed by ice heated from −5 to 0°C:

$C_{ice}\, m_{ice}\, \Delta T_1 = 2000 \cdot 0.14 \cdot 5 = 1400\ J$

6 points Heat absorbed by melting ice

$\lambda\, m_{ice} = 334 \times 10^3 \cdot 0.14 = 46760\ J$

6 points heat produced by cooling the water to temp 0°C

$C_w\, m_w\, \Delta T_2 = 4190 \cdot 0.2 \cdot 40 = 33520\ J$

7 points enough to heating the ice, but not enough for melting all the ice → final temparatare = <u>0°C</u>.

Thermo A3

7 points (a) $p_1 V_1^\gamma = p_2 V_2^\gamma \qquad \gamma = \frac{C_v + R}{C_v} = \frac{5/2}{3/2} = 5/3$

$$p_2 = p_1\left(\frac{V_1}{V_2}\right)^\gamma = 1\times10^5\left(\frac{0.1}{0.06}\right)^{5/3} = 2.34\times10^5\,Pa$$

6 points (b)

$$W = \int_{V_1}^{V_2} p\,dV = p_1 V_1^\gamma \int_{V_1}^{V_2} \frac{dV}{V^\gamma} = \frac{p_1 V_1^\gamma}{\gamma-1}\left[\frac{1}{V_1^{\gamma-1}} - \frac{1}{V_2^{\gamma-1}}\right]$$

$$= \frac{p_1 V_1}{\gamma-1}\left[1-\left(\frac{V_1}{V_2}\right)^{\gamma-1}\right] = \frac{10^5\cdot 0.1}{2/3}\left[1-\left(\frac{0.1}{0.06}\right)^{2/3}\right] = -6086\,J$$

"–" means that work is done <u>on</u> the gas

6 points → (c) $Q = 0$ (adiabatic process)

6 points → (d) $\Delta U = -W = 6086\,J$

Thermo A4

10 points

$$m_1 c(T - T_1) = m_2 c(T_2 - T) \qquad T_1 = 10°C = 283K$$
$$T_2 = 80°C = 353K$$

$$T = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} = \frac{3\cdot 10 + 1\cdot 80}{4} = 27.5(°C) = 300.5K$$

15 points

$$\Delta S = m_1 c \int_{T_1}^{T} \frac{dT}{T} + m_2 c \int_{T_2}^{T} \frac{dT}{T} = c\left(m_1 \ln\frac{T}{T_1} + m_2 \ln\frac{T}{T_2}\right)$$

$$= 4190\left(3 \ln\frac{300.5}{283} + 1\cdot \ln\frac{300.5}{353}\right) = 79.5\,\frac{J}{K}$$

Thermo B1

8 points

(1) heat released:

$$\Delta Q_1 = \lambda m_{steam} + c_{wat} m_{steam} \Delta T_1$$

where $m_{steam} = [0.525 - (0.150 + 0.340)]\,kg = 0.035\,kg$

$$\Delta T_1 = (100 - 71)\,K = 29\,K$$

8 points

(2) heat absorbed

$$Q_2 = [c_{cal} m_{cal} + c_{water} m_{water}] \Delta T_2$$

$$\Delta T_2 = (71 - 15)\,K = 56\,K$$

9 points

solve $Q_1 = Q_2 \rightarrow$

$$\lambda m_{st} + c_{wat} m_{st} \Delta T_1 = [c_{cal} m_{cal} + c_{wat} m_{wat}] \Delta T_2$$

$$\lambda = \frac{[c_{cal} m_{cal} + c_{wat} m_{wat}] \Delta T_2 - c_{wat} m_{st} \Delta T_1}{m_{st}}$$

$$= \frac{(420 \cdot 0.15 + 4190 \cdot 0.34) \cdot 56 - 4190 \cdot 0.035 \cdot 29}{0.035}$$

$$= 2258 \times 10^3 \, \frac{J}{kg}$$

THERMO SHORT A1

Consider a heat engine which does 2000 BTU of work for every 2500 BTU of heat it absorbs. If the heat sink for the device is a thermal bath of ice water (0°C), what would be the minimum source temperature in °C?

(The BTU, or British Thermal Unit, is a unit of energy equal to 1054 joules)

Answer

10 points

The efficiency of the device is $\eta = \frac{W}{Q_{\text{in}}} = \frac{2000 \text{ BTU}}{2500 \text{ BTU}} = 80\%$. Note that there is no need to convert BTU into other units.

10 points

The maximum possible efficiency is $\eta_{\max} = 1 - \frac{T_{\text{lo}}}{T_{\text{hi}}}$ (the efficiency of the Carnot engine).

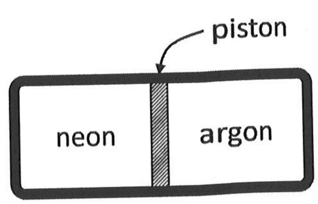
5 points

Setting the two equal gives $1 - \frac{T_{\text{lo}}}{T_{\text{hi}}} = 0.80 \quad \Rightarrow \quad T_{\text{hi}} = \frac{T_{\text{lo}}}{0.20} = 5T_{\text{lo}} = 5 \times 273 = 1365 \text{ K} = 1092°\text{C}$

B2

THERMO LONG

A cylindrical container is initially separated by a clamped piston into two compartments of equal volume. The left compartment is filled with one mole of neon gas at a pressure of 4 atmospheres and the right with argon gas at one atmosphere. The gases may be considered as ideal. The whole system is initially at temperature T = 300 K, and is thermally insulated from the outside world. The heat capacity of the cylinder-piston system is C (a constant).



The piston is now unclamped and released to move freely without friction. Eventually, due to slight dissipation, it comes to rest in an equilibrium position. Calculate:

(a) The new temperature of the system (the piston is thermally conductive).
(b) The ratio of final neon to argon volumes.
(c) The total entropy change of the system.
(d) The additional entropy change which would be produced if the piston were removed.
(e) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (a), (b), (c), and (d) would be different?

Answers

5 points

(a) The internal energy of an ideal gas is a function dependent only on temperature, so the internal energy of the total system does not change. Neither does the temperature. The new equilibrium temperature is 300 K.

5 points

(b) The volume ratio is the ratio of molecular numbers, and is also the ratio of initial pressures. Thus, $\frac{V_{Ne}}{V_{Ar}} = \frac{4}{1} = \frac{1}{n}$, where $n = \frac{1}{4}$ is the number of moles of the argon gas.

5 points

(c) The increase of entropy of the system is

$$\Delta S = n_{Ne} R\ln\left(\frac{V_2}{V_1}\right)_{Ne} + n_{Ar} R\ln\left(\frac{V_2}{V_1}\right)_{Ar} = R\ln\left(\frac{4/5}{1/2}\right) + \frac{1}{4} R\ln\left(\frac{1/4}{1/2}\right) = 2 \text{ J/K}$$

1/5

typo

5 points

(d) The additional entropy change is

$$\Delta S' = R\ln(1+n) + nR\ln\left(\frac{1+n}{n}\right) = 5.2 \text{ J/K} \quad (\text{note } n = 1/4, \text{ see part a})$$

(e) If initially the gas on the left is a mole of argon, the answers to (a), (b), and (c) will not change. As for (d), we now have $\Delta S' = 0$.

5 points

B3 solution

One mole of real gas that follows the equation of state: $p(V-b) = RT$, where $0<b<V$ is a small constant. Derive the work done by the gas after isothermal expansion from V to V_f. Compared with the work done by ideal gas after isothermal expansion from V to V_f, which work is larger?

Solution:

For this real gas, $W= \int PdV = \int RT/(V-b)\, dV = RT \ln[(V_f-b)/(V-b)]$. ← 9 points

For ideal gas, $W= \int PdV = \int RT/V\, dV = RT \ln(V_f/V)$. ← 9 points

Since $V_f/V<(V_f-b)/(V-b)$, the real gas does more work. ← 7 points

B4 solution

Consider boiling water into water vapor in the atmosphere. (a) How much is the work done by the water per kg? (b) What's the change of internal energy per kg?

Data you may need: the specific volume (volume per unit mass) is $V_M^w = 1.0\times10^{-3}$ m^3/kg and $V_M^v = 1.8$ m^3/kg for water and water vapor respectively. The pressure of atmosphere is 1.01×10^5 Pa. The latent heat of the liquid to vapor transition is 334 kJ/kg.

Solution:

The work done can be calculated by

$$\left.\begin{aligned} W &= P\Delta V = P(V_{Mv}-V_{Mw}) \\ &= 1.01E5 * (1.8 - 1\times10^{-3})m3/kg \\ &= 180\ kJ/kg \end{aligned}\right\} \text{8 points}$$

The latent heat $\Delta H = \Delta U + P\Delta V = \Delta U + W$ ← 9 points

$\Delta U = \Delta H - W = 334 - 180 = 154$ kJ/kg. ← 8 points