UNL - Department of Physics and Astronomy

# Preliminary Examination - Day II
# Friday, May 23, 2025

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 “A” questions and 4 “B” questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

*Note*: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

**WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY**

**Thermodynamics and Statistical Mechanics Group A**

*Answer only two Group A questions*

**A1.** For a wire with equation of state $F = bT\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right)$, where $L$ is the length, $L_0(T)$ is the length when the tension $F$ is zero, $b$ is a constant. Calculate the work done by the environment when the length changes from $L_0$ to $\frac{L_0}{2}$ at constant temperature.

**A2**. At low temperature, the constant volume molar specific heat of a solid is $C_v = a\left(\frac{T}{\theta_D}\right)^3$, where $\theta_D$ is called Debye temperature. Consider a solid with $a$ = 1.94 kJ/mol/K and $\theta_D$ = 27 ºC, calculate the heat per mole the solid needs to absorb to increase temperature from 5 K to 10 K.

**A3.** Calculate $C_P = \left(\frac{\partial H}{\partial T}\right)_P$ using the chain rule for a simple solid for which $H(T, V) = Mc_0T + (a_0T - b_0)V$ and $V = V_0 \exp[(a_0T - P)/b_0]$. Show that the result is $C_P = Mc_0 + \frac{a_0^2}{b_0}TV$.

**A4.** A compressor designed to compress air is used instead to compress helium. It is found that the compressor overheats. Explain this effect, assuming that the compression is approximatively adiabatic, and the starting pressure is the same for both gases. For air, $\gamma = 5/3$; and for helium, $\gamma = 7/5$.

**Thermodynamics and Statistical Mechanics Group B**

*Answer only two Group B questions*

**B1.** For ideal gas, if the heat capacity of a process is constant, then the process is polytropic $PV^l = constant$. Assuming that $C_p$ and $C_v$ are constants, find *l*.

**B2.** In the Einstein model of specific heat for diatomic molecules, atomic vibrations can be treated as independent quantum oscillators with quantized energy: $\mathcal{E}_\mathrm{n} = \mathrm{n}\hbar\omega$ and the population of excited oscillators follow the Boltzmann distribution.

a) At thermal equilibrium, show that the average energy for the quantum oscillators is given by: $\bar{\mathcal{E}} = \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}$.

Find expression for the specific heat $C_v$ of 1 mole of the diatomic molecules. Show that it approaches the 3$R$ limit at high temperatures, where $R = N_A k_B$ is the universal gas constant

**B3.** A solid with heat capacity $C_A$ at temperature $T_A$ is placed in contact with another solid with heat capacity $C_B$ at a lower temperature $T_B$. What is the change in entropy of the system after the two bodies have reached thermal equilibrium?

**B4.**

(a) How much heat is required to raise the temperature of $1000$ grams of nitrogen from $-20^oC$ to $100^oC$ at constant pressure?

(b) How much the internal energy of the nitrogen increased?

(c) How much external work was done?

(d) How much heat is required if the volume is kept constant?

Take the specific heat at constant volume $c_v = 5$ cal/mole °C and $R = 2$ cal/mole °C.

## Classical Mechanics Group A

*Answer only two Group A questions*

**A1.** A playground merry-go-round has a radius 2.4 m and a moment of inertia 2100 kg·m$^2$ about a vertical axis through its center and turns with negligible friction. a) A child applied an 18 N force tangentially to the edge of the merry-go-round for 15 s. If the merry go-round is initially at rest, what is the angular speed after the 15 s interval ? b) How much work did the child do on the merry-go-round ?

**A2.** A hoop of radius 0.10 and mass of 0.20 kg rolls down an inclined plane without slipping. It starts from the rest at a height of 2.0 m.

(a) What is the total kinetic energy at the bottom of the inclined plane?
(b) What is the rotational kinetic energy of the hoop at the bottom?
(c) What is the translational kinetic energy at the bottom?
(d) What is the angular velocity at the bottom?

**A3.** An object of mass $m_1$ moving with velocity $v_{1i}$ collides head-on with an object of mass $m_2$ at rest. Suppose the two objects stick together after the collision and now move together with a velocity $v_f$. Show that this type of collision is always exoergic by determining the transferred energy quantity, $Q = T_i - T_f$, from the initial and final kinetic energies. Do your calculation in both the lab frame and center-of-mass frame. Does the value of $Q$ dependent on the reference frame?

**A4.** A mass $m$ sliding horizontally is subject to a viscous quadratic drag force $F = -c\dot{x}^2$. It is initially moving with a velocity $v_0$ at the origin. Find the velocity $v(x)$ as a function of the displacement $x$ and show that the mass never comes to rest.

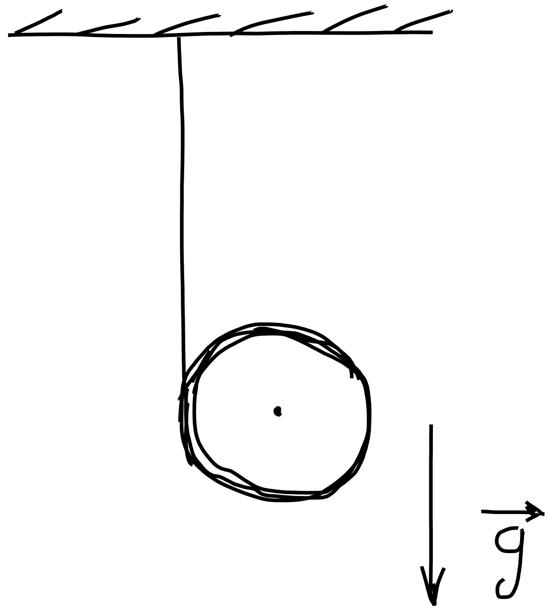
## Classical Mechanics Group B

*Answer only two Group B questions*

**B1.** A concrete board is leaned up against a wall with a 60$^o$ angle between the board and the floor. It weighs 70 kg.

(a) Calculate the minimum possible coefficient of static friction such as no slipping occurs(b) Calculate the force of friction between the ground and the board;

**B2.** A string is wrapped around a hoop of radius 3 cm and mass 20 g which falls under gravity




(a) What is the angular acceleration of the hoop?
(b) Calculate the time it takes for the hoop to descend 0.5 m;
(c) Calculate the angular velocity of the rotating hoop after it has descended 0.5 m.

**B3.** Two balls of masses $m_1$ and $m_2$ with velocities $v_1$ and $v_2$ collide elastically head-on. Find their velocities after the collision. Analyze the case when (a) $m_1=m_2$, and (b) $m_1>>m_2$.

**B4**. A ballistic projectile of mass $m = 50$ grams is launched from a vehicle. The initial speed $v$ of the projectile is 500 m/s southward, at an angle $\theta$ of $\pi/6$ from the horizontal. Neglect the motion of the Earth.

(a) Neglecting air resistance, show that the projectile trajectory is a parabola and calculate the horizontal range $R$ of the projectile when the vehicle is initially stationary.
(b) If there is now a non-negligible linear air resistance, $\mathbf{F}=-c_1\mathbf{v}$ with $c_1=6.65\times10^{-5}$ kg/s , find the new horizontal range $R'$.

Help for part (b): For $|u| < 1$, use the relation: $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots$

**Physical Constants**

Speed of light ...................... $c = 2.998 \times 10^{8}$ m/s
Atmospheric pressure........... $101,325$ Pa
Electron mass ...................... $m_e = 9.109 \times 10^{-31}$ kg
Avogadro constant .............. $N_\mathrm{A} = 6.022 \times 10^{23}$ mol$^{-1}$
Boltzmann constant.............. $k_\mathrm{B} = 1.381 \times 10^{-23}$ J/K = $8.617 \times 10^{-5}$ eV/K
Gas constant ......................... $R = 8.314$ J/(mol·K)
Atomic mass unit ................ $1\,\mathrm{u} = 1.66 \times 10^{-27}$ kg
Gravitational constant ........... $G = 6.674 \times 10^{-11}$ m$^3$/(kg·s$^2$) ; $g = 9.8$ m/s$^2$

**Equations That May Be Helpful**

**<u>TRIGONOMETRY</u>**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small $x$:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

**THERMODYNAMICS**

Heat capacity $C_V = N\dfrac{d\langle E\rangle}{dT}$ .

Molar heat capacity of diatomic gas: $C_V = \frac{5}{2}R$ .

For adiabatic processes in an ideal gas with constant heat capacity, $pV^\gamma = \text{const}$ .

$$dU = TdS - pdV$$
$$dF = -SdT - pdV$$

$$H = U + pV$$
$$F = U - TS$$
$$G = F + pV$$
$$\Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$
$$C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$
$$TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$
$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$

Carnot engine: $\Delta S$=0

Carnot efficiency = 1 – $T_c/T_h$ .

The cyclic rule: $\left(\frac{\partial T}{\partial P}\right)_H \left(\frac{\partial P}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_P = -1.$

Stefan-Boltzmann's law:

$$P = \sigma T^4; \quad \sigma = 5.67\times10^{-8} W\cdot m^{-2}\cdot K^{-4}$$

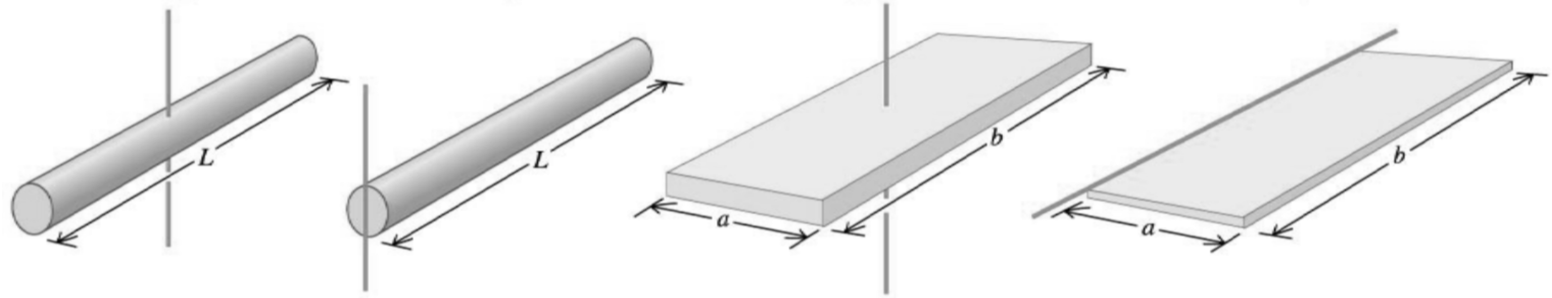
**TABLE 9.2 Moments of Inertia of Various Bodies**

(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$



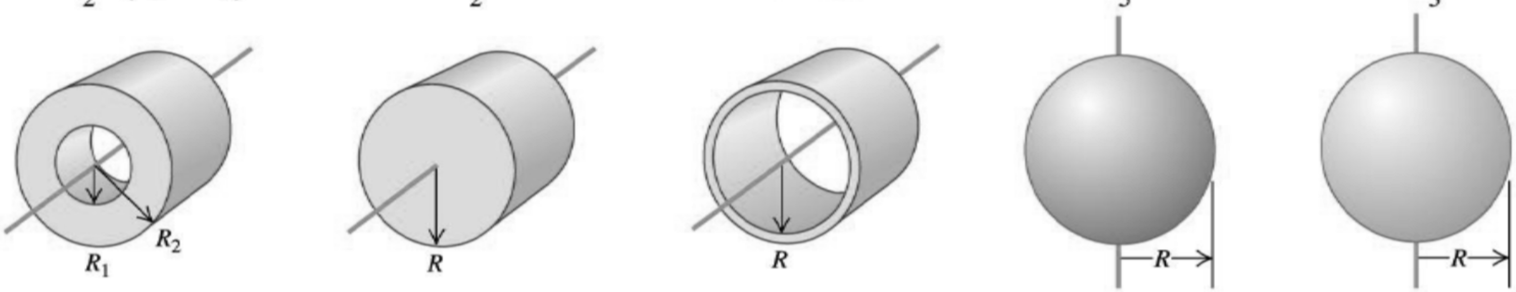

(e) Hollow cylinder
$I = \frac{1}{2}M(R_1^2 + R_2^2)$

(f) Solid cylinder
$I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder
$I = MR^2$

(h) Solid sphere
$I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$



## VECTOR DERIVATIVES

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

*Gradient*: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence*: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

*Curl*: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

*Laplacian*: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient*: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence*: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

*Curl*: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian*: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

*Gradient*: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence*: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

*Curl*: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

*Laplacian*: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

## VECTOR IDENTITIES

### Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

### Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

### Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem**: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem**: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem**: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

## CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

## INTEGRALS

| | |
|---|---|
| $\int_0^\infty \frac{1}{1+bx^2}dx = \frac{\pi}{2b^{1/2}}$ | $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$ |
| $\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$ | $\int_0^\infty xe^{-x^2}dx = \frac{1}{2a}$ |
| $\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$ | $\int_0^\infty x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{2a^{3/2}}$ |
| $\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$ | $\int_0^\infty x^3e^{-x^2}dx = \frac{1}{2a^2}$ |
| $\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$ | $\int_0^\infty x^4e^{-x^2}dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$ |
| $\int (x^2+b^2)^{-2}dx = \frac{\frac{bx}{x^2+b^2}+\arctan\left(\frac{x}{b}\right)}{2b^3}$ | $\int_0^\infty x^5e^{-x^2}dx = \frac{1}{a^3}$ |
| $\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$ | $\int_0^\infty x^6e^{-x^2}dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$ |
| $\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$ | |
| $\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) =$ | |
| $= -\frac{1}{ab}\,\mathrm{artanh}\left(\frac{ax}{b}\right)$ | |