UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Tuesday, May 18, 2021

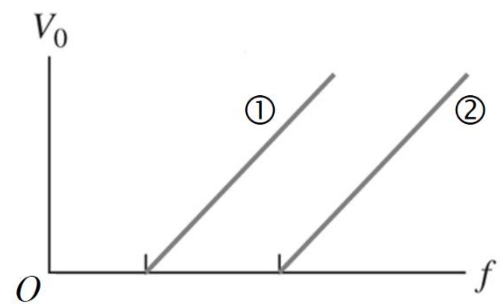
This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

<u>Quantum Mechanics Group A</u> <u>- Answer only two Group A questions</u>

A1. Photoelectric experiments are done with the target materials ① and ② with work functions ϕ_1 and ϕ_2, respectively. For each material, the stopping potential V_0 is plotted as a function of the frequency f of the light used. The two straight lines in the adjacent graph show the result.

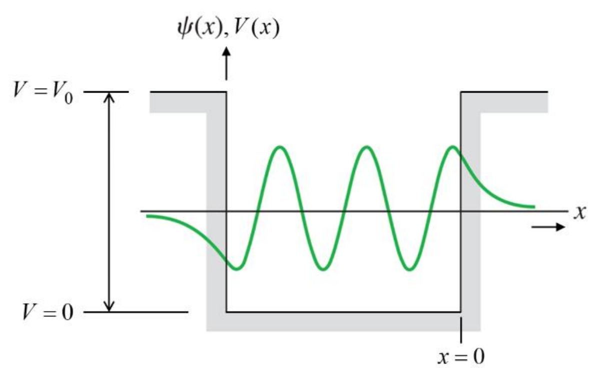


a. Explain in a few sentences what "stopping potential" means.
b. What is the slope of the straight lines?
c. Which material has the higher work function, ① or ②?

The work function of material ① is $\phi_1 = 4.30\ \text{eV}$.

d. What is the largest wavelength light can have to cause photoemission from material ① ?

A2. In this one-dimensional problem, we consider a stationary state of an electron with energy E in a finite well of depth V_0. The adjacent diagram shows both the potential $V(x)$ and the electron's wave function $\psi(x)$ as a function of position x. The electron is in a bound state, meaning $E < V_0$. We will focus on the part of the wave function in the classically forbidden region $x > 0$.



a. Why is this region called "classically forbidden"?
b. Show that the wave function $\psi(x) = Ae^{-\kappa x}$ (with $\kappa > 0$ and A some constant) satisfies the time-independent Schrödinger equation in this region, and derive an expression for κ.
c. The wave function $\psi(x) = Be^{+\kappa x}$ also satisfies the Schrödinger equation in this region, but must be rejected. Why?

At $x = 0$, we have $\psi(0)$. At a certain position $x_0 > 0$, the value of the wave function has dropped to $\alpha\psi(0)$, with $\alpha < 1$.

d. Find a relationship between κ, x_0, and α.
e. Use your answer of part *d*. and the expression you found for κ in part *b*. to find an expression for V_0.
f. If $E = 2.27\ \text{eV}$, $x_0 = 1.1\ \text{Å}$, and $\alpha = 0.13$, find the depth of the well in eV.

A3. In this one-dimensional problem, a particle of mass *m* is inside a potential well given by

$$U(x) = U_0 \cosh(bx)$$

When the particle is not far from the equilibrium position at $x = 0$, the potential $U(x)$ may be approximated by a parabolic potential.

a. Calculate the parabolic potential as a function of x.

Close to the equilibrium distance, we may consider the system as a harmonic oscillator.

b. Find the spring constant of this harmonic oscillator.

We have $U_0 = 10.0 \text{ eV}$, $b = 2.00 \times 10^9 \text{ m}^{-1}$, and $m = 9.11 \times 10^{-31} \text{ kg}$.

c. Calculate $\hbar\omega$ in joules and in eV.

d. The system makes a transition from the state with *n* = 3 to the state with *n* = 1, emitting a single photon. Calculate the wavelength of this photon in nanometers.

A4. Consider a spin-$\frac{7}{2}$ particle (i.e. it has *s* = $\frac{7}{2}$).

a. What is the magnitude of its spin vector $\vec{\boldsymbol{S}}$?

b. Can the spin vector of this particle be *perpendicular* to the *z*-axis?

c. Calculate the smallest angle the spin vector can make with the positive *z*-axis.

<u>Quantum Mechanics Group B</u> <u>- Answer only two Group B questions</u>

B1. The operators $\hat{L}$ and $\hat{M}$ are Hermitian (self-adjoint). Operators $\hat{A}$ and $\hat{B}$ are not.

a. Is $\exp(\hat{L})$ Hermitian?

b. Is $[\hat{L},\hat{M}]$ Hermitian?

c. Is $[\hat{A}^{\dagger},\hat{B}]+[\hat{A},\hat{B}^{\dagger}]$ Hermitian?

B2. A harmonic oscillator (mass *m*, frequency ω) is in the initial state

$$\psi(x,0) = \frac{1}{\sqrt{2}}[\varphi_0(x) + \varphi_1(x)]$$

where φ_0, φ_1 are the energy eigenstates.

(a) Calculate the expectation value of *x* as a function of time. You can use

$$\langle\varphi_1|x|\varphi_0\rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

(b) Calculate *x*(*t*) for a classical oscillator with the initial condition $x(0)=x_0, v(0)=0$. Express x_0 in terms of the oscillator energy *E*, *m* and ω, and compare *x*(*t*) with the quantum result. What value should *E* have in order for the classical and quantum results to coincide?

B3. A free particle of mass *m* is moving in one dimension. At *t*=0 its wavefunction is

$$\psi(x,0) = \sin(k_1 x) + 2\cos(k_2 x)$$

(a) Find $\psi(x,t)$.
(b) At the time *t* the momentum *p* is measured. What are possible outcomes of this measurement and what are their probabilities?
(c) Suppose that *p* is measured at *t*=1 s and the value $\hbar k_1$ is found. What is $\psi(x,t)$ at *t*>1 s? Is the particle's state at *t*>1 s an energy eigenstate? If yes, what is the energy eigenvalue? If not, what is the energy expectation value?

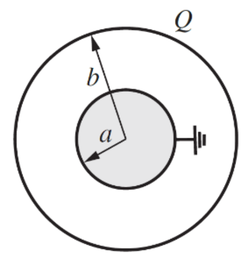
B4. An electron is prepared in the state which is an eigenstate of S_z with the eigenvalue $\hbar/2$, and then S_x is measured.

(a) What are possible outcomes of this measurement and what are the corresponding probabilities?
(b) What are the expectation values of S_z and S_x for the initial state?
(c) Suppose an electron is placed in a uniform magnetic field of 1 T. Its spin flips, and as a result, electron emits a photon. Find the photon's frequency.

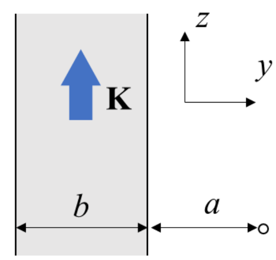
<u>Electrodynamics Group A</u> <u>- Answer only two Group A questions</u>

A1. A metal hollow sphere of radius *R* is kept under a constant potential Φ_0. Using Gauss's law, find the electric field **E** and the electrostatic potential Φ inside and outside the sphere and determine the surface charge density σ.

A2. A spherical conducting shell of radius *b* is concentric with and encloses a conducting ball of radius *a*. The ball is grounded, and the shell has charge *Q*. Argue that the presence of the charge *Q* on the shell will draw up a charge onto the ball from ground. Find the magnitude of this charge.



A3. A strip of width *b* carries a uniform surface current $\mathbf{K} = K\hat{\mathbf{z}}$. Find the magnetic field **B** at a point in the plane of the strip that lies at perpendicular distance *a* from the strip in the $\hat{\mathbf{y}}$-direction.

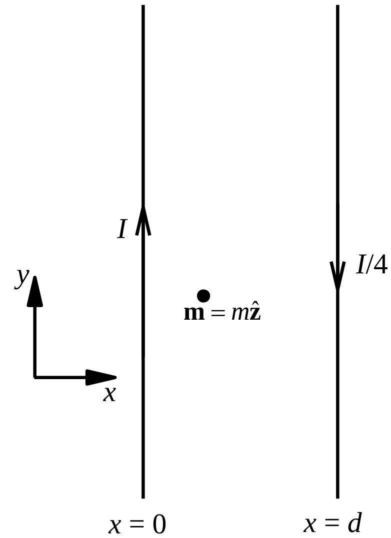


A4. In a proton supercollider, the protons in a 5-mA beam move with nearly the speed of light.

1. How many protons are there per meter of the beam?
2. What is the average separation of the protons, if the cross-sectional area of the beam is 10^{-6} m^2?

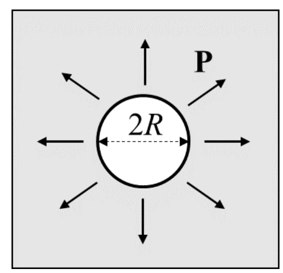
<u>**Electrodynamics Group B** - Answer only two Group B questions</u>

B1. Two infinite parallel wires are oriented along the *y*-direction and placed at a distance *d* apart. One wire carries a current I, and the other carries current $I/4$ in the opposite direction. Between these two wires there is a particle with position constrained to be in the plane formed by the two wires (the *x*-*y* plane). This particle has magnetic moment, **m**, that is *fixed* in magnitude and direction along +*z*, i.e. $\mathbf{m} = m\hat{\mathbf{z}}$.



1. Find the equilibrium position, *x*, between the two wires ($0 < x < d$) of the particle that minimizes the interaction energy of this magnetic moment with the fields generated by the currents of the wires.
2. Suppose the direction of the magnetic moment is not fixed. Is the position you found in part (1) a real equilibrium?

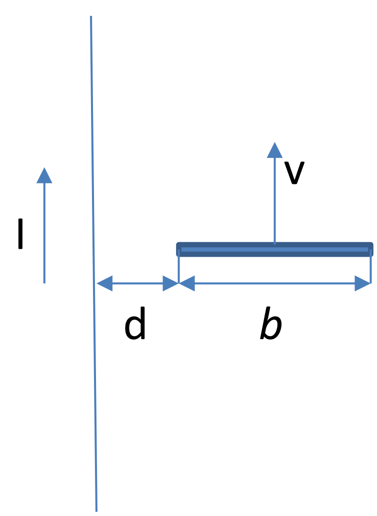
B2. A polarized volume with a radial distribution of polarization, i.e. $\mathbf{P} = P\hat{\mathbf{r}}$, where *P* is constant, has a small spherical hole of radius *R* at the origin. Find the polarization induced bound charge density and the electric field everywhere. You may assume that the volume of polarized material is far larger than that of the spherical hole.



B3. A coil with inductance 4 mH and resistance 150 Ω is connected across a battery of emf 12 V and negligible internal resistance.

a. What is the initial rate of increase of the current?
b. What is the rate of increase when the current is half its final value?
c. What is the final current?
d. How long does it take for the current to reach 99% of its final value?

B4. A conducting rod of length *b* lies with its length perpendicular to a long wire carrying current *I*, as shown in the figure. The near end of the rod is at a distance *d* away from the wire. The rod moves with a speed v in the direction of the current.



(a) Find the potential difference *V* between the ends of the rod.

(b) Interpret the sign of *V*.

Physical constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant .. $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge ..... $e = 1.602 \times 10^{-19}$ C

electric permittivity ... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant...... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic const.... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy 511.0 keV

Compton wavelength $\lambda_C = h / m_{el} c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / k e^2 m_{el} = 0.5292$ Å

1 hartree (= 2 ryd) $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

Stefan-Boltzmann constant σ=5.67x10^{-8} W/(m^2 K^4) fine structure constant $\alpha = ke^2/(\hbar c)$

Gyromagnetic ratio of electron: $|\gamma_{el}| = 1.761 \times 10^{11}$ C/kg $\approx e / m_{el}$

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$\cos(ix)=\cosh(x)$

$\sin(ix)=i\sinh(x)$

For small x:
$\sin x \approx x - \frac{1}{6}x^3$
$\cos x \approx 1 - \frac{1}{2}x^2$
$\tan x \approx x + \frac{1}{3}x^3$

QUANTUM MECHANICS

$$\left[AB,C\right] = A\left[B,C\right] + \left[A,C\right]B$$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_+ \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell, m+1\rangle$$
$$L_- \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell, m-1\rangle$$

Creation, annihilation operators:

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle \qquad \hat{a}\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle$$

Probability current density: $J(x) = \dfrac{\hbar}{2mi}\left(\psi^*\dfrac{\partial\psi}{\partial x} - \psi\dfrac{\partial\psi^*}{\partial x}\right) = \dfrac{\hbar}{m}\mathrm{Im}\left(\psi^*\dfrac{\partial\psi}{\partial x}\right)$.

$$H_{\mathrm{mag}} = -\gamma\,\mathbf{S}\cdot\mathbf{B}$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

ELECTROSTATICS

$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0}$ $\qquad \mathbf{E} = -\nabla V$ $\qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$ $\qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau = Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$, in which

$q = \int \rho(\mathbf{r}) d^3\mathbf{r}$ is the monopole moment

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r} d^3\mathbf{r}$ is the dipole moment

$Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right] d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$)

Relative permittivity: $\varepsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}}$

Bound charges

$\rho_{\mathrm{b}} = -\nabla\cdot\mathbf{P}$

$\sigma_{\mathrm{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0\frac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0\frac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0\frac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

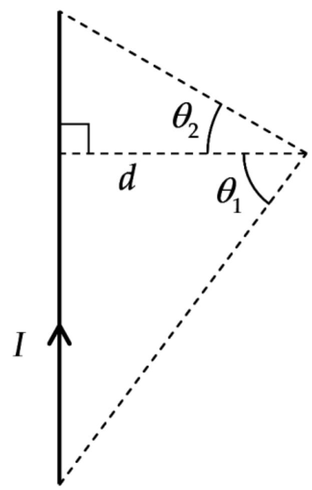
Relative permeability: $\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}}$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I d\boldsymbol{\ell} \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl}}$



$$B = \frac{\mu_0 I}{4\pi d}\left(\sin\theta_1 + \sin\theta_2\right)$$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$
Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}$

Bound currents

$J_b = \nabla \times \mathbf{M}$
$K_b = \mathbf{M} \times \hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \dfrac{\partial \mathbf{D}}{\partial t}$ Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I} d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl :

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi/2b^{1/2}$$
$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$
$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$