UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, May 26, 2023

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1. The photon of 200 nm wavelength is completely absorbed by a copper frying pan, and transfers all its momentum to the frying pan.

(a) What is the momentum given to the frying pan by this photon?
(b) The photon causes a photoelectron to be ejected from the frying pan along the same line as the incident photon but in the opposite direction. The copper frying pan has a work function of 4.3 eV. What is the resulting momentum of the frying pan?

A2. An operator Q is called anti-hermitian when it equals *minus* its hermitian conjugate: $Q = -Q^{\dagger}$.

a. Show that the commutator of two hermitian operators is anti-hermitian.
b. Is the commutator of two *anti*-hermitian operators also anti-hermitian?
c. Show that the eigenvalues of an anti-hermitian operator are purely imaginary.

A3. For what de Broglie wavelength is the kinetic energy K of an electron equal to the energy of a photon with wavelength 1 nm?

Is using the nonrelativistic expression for K justified? Give a quantitative estimate.

A4. A free particle of mass m is moving in one dimension. At t=0 the particle's wavefunction is

$$\psi(x,0)=\sin(kx).$$

(a) Suppose at t=0 the particle's momentum is measured. What is (are) the outcome(s) of this measurement and what is (are) the corresponding probability(ies)? Write down the particle's wave function(s) immediately after the measurement. Don't worry about the normalization.
(b) Suppose at t=0 the particle's parity is measured. Answer the same questions as in part (a).
(c) If no measurements are made, what is the particle's wave function at t>0?

Quantum Mechanics Group B - Answer only two Group B questions

B1. Consider the one-dimensional wave function $\psi(x) = A(x/x_0)^n e^{-x/x_0}$, where A, n and x_0 are constants.

a) Use the stationary Schrödinger equation to find the potential $V(x)$ and energy E for which this wave function is an eigenfunction. Assume that as $x \to \infty$, $V(x) \to 0$.
b) What connection do you see between this potential and the effective radial potential for a hydrogenic state of orbital angular momentum l?

B2. A charged particle is moving in a magnetic field $\vec{B}$. Find the commutation rules for the Cartesian components of the particle's velocity.

B3. Consider an electron in a uniform magnetic field *B* pointing along the *z* direction. At time *t* = 0, the electron spin is along the positive *y* direction, with spin state vector

$$|\psi\rangle = \tfrac{1}{2}\sqrt{2}\left(\ |\uparrow\rangle\ +\ i|\downarrow\rangle\ \right) = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

a. Show that $|\psi\rangle$ is an eigenstate of S_y, and give the eigenvalue.
b. Find the spin state vector for $t > 0$.
c. Find the expectation values of S_z and S_x for $t > 0$.

B4. The wave function of a harmonic oscillator (mass *m*, frequency ω) at *t*=0 is given by a superposition of the normalized ground state and first excited state:

$$\psi(x) = C\left[\varphi_0(x) + \varphi_1(x)\right] .$$

a. Find the normalization constant *C* (a positive real-valued number).
b. Find the expectation value of *x* at *t*=0.
c. Find the wave function and the corresponding probability density for *t*>0.
d. Find the expectation value of *x* as a function of time.

Electrodynamics Group A - Answer only two Group A questions

A1. An infinite plane of a uniform surface charge density σ_0 is placed at distance $z = h$ above the surface of a half-space grounded metal.

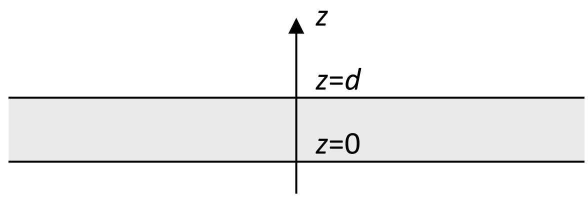
(a) Find the potential and the electric field in the whole space.

(b) Find the induced surface charge density on a metal surface.

A2. A positive charge is distributed over a thin metal plate of infinite area and thickness *d* in such a way that there is charge *Q* per area *A* of the plate. Assuming that the bottom surface of the plate lies at *z* = 0 and the top surface lies at *z* = *d*,

(a) Find the distribution of charge across the plate (along the *z* direction) and write the expression for the volume charge density in all space;

(b) Find and sketch the electric field and the electrostatic potential along the *z* direction.



A3. A conducting rod of mass m and resistance R is free to slide without friction along two parallel rails of negligible resistance. The rails are separated by a distance *l* and inclined at an angle θ to the horizontal. Another conducting rod of negligible resistance is placed at the base of the rails making a closed circuit. There is a magnetic field *B* directed upward. Find the terminal speed of the rod. (Assume that the rails are long enough for this).

A4. A large electromagnet has an inductance 40 H and a resistance 10 Ω. It is connected to a dc power source of 250V. Find the time for the current to reach 10 A.

Electrodynamics Group B - Answer only two Group B questions

B1. A flat surface z=0 of a semiinfinite linear dielectric material of uniform dielectric permittivity ε is affected by an external non-uniform electric field which is directed perpendicular to the surface and whose magnitude in the absence of the dielectric is $E^{ext}(\mathbf{r})$. There are no free charges in the dielectric.

(1) Show that the volume polarization charge density in the dielectric is zero;

(2) Show that the normal component of the polarization-induced electric field $E_z^P(\mathbf{r})$ near the surface inside the dielectric is given by $E_z^P(\mathbf{r}) = -\dfrac{\sigma_P(\mathbf{r})}{2\varepsilon_0}$ where $\sigma_P(\mathbf{r})$ is the induced surface polarization charge density.

(3) Express $\sigma_P(\mathbf{r})$ is terms of the total electric field at the surface;

(4) Express $\sigma_P(\mathbf{r})$ in terms of $E^{\text{ext}}(\mathbf{r})$.

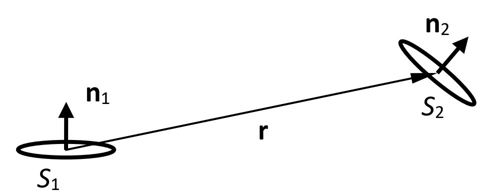
B2. A linearly polarized electromagnetic wave plane wave of frequency ω, traveling in vacuum in the *z* direction and polarized in the *x* direction, is reflected from a perfect conductor (i.e., a conductor with infinite conductivity σ) whose surface lies in the *x*-*y* plane. Write expressions for the incident, transmitted, and reflected waves and using the boundary conditions for the electric and magnetic fields show that the reflection coefficient is equal to 1.

B3. Two small planar loops of wire of area S_1 and S_2 have normal unit vectors to their planes, $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively, and are separated by radius vector $\mathbf{r}$, large compared to the size of the loops.

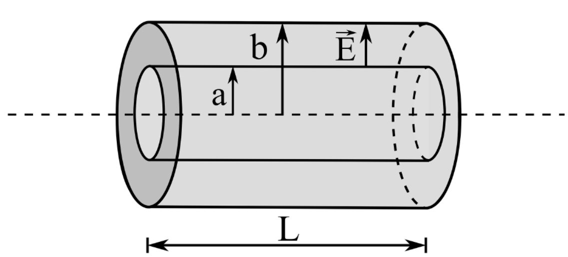
(a) Find the mutual inductance of the system.

(b) Assuming that $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel find the angle between $\mathbf{r}$ and $\mathbf{n}_1$ ($\mathbf{n}_2$) at which the mutual inductance is zero.

(c) Sketch magnetic field lines with respect to the current loops, explaining the result (b).



B4. Most materials are neither perfect conductors nor perfect dielectrics. Many capacitors with dielectrics will "leak" a small current. Consider a cylindrical coaxial capacitor with a leakage current between the two long electrodes of radii *a* and *b*, with *b* > *a*, separated by an unknown material of conductivity *σ* and permittivity close to ε_0. They are maintained at an electric potential *V* with a current *I* flowing from one to another in a length *L*. Find the conductivity *σ*.



Physical constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant .. $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge ..... $e = 1.602 \times 10^{-19}$ C

electric permittivity ... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant...... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

fine structure constant $\alpha = ke^2/(\hbar c)$

electrostatic const.... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy 511.0 keV

Compton wavelength $\lambda_C = h/m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 Ry) $E_h = \hbar^2/m_{el}a_0^{\,2} = 27.21$ eV

gravitational constant $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

1 Ry =13.6 eV

Equations That May Be Helpful

TRIGONOMETRY

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$
$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$\cos(ix) = \cosh(x)$
$\sin(ix) = i\sinh(x)$

For small x:
$\sin x \approx x - \tfrac{1}{6}x^3$
$\cos x \approx 1 - \tfrac{1}{2}x^2$
$\tan x \approx x + \tfrac{1}{3}x^3$

QUANTUM MECHANICS

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_+ \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell, m+1\rangle$$
$$L_- \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell, m-1\rangle$$

Gyromagneic ratio for electron (SI units) = e/m

Pauli matrices:
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z) = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\,e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z) = \mp\sqrt{\frac{3}{8\pi}}\,\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$	$Y_{20}(x,y,z) = \sqrt{\frac{5}{16\pi}}\,\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\,e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z) = \mp\sqrt{\frac{15}{8\pi}}\,\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\,e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z) = \mp\sqrt{\frac{15}{32\pi}}\,\frac{x^2-y^2\pm 2ixy}{r^2}$

Stationary states of harmonic oscillator for n=0 and n=1

$$\varphi_0(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\alpha^2 x^2/2}$$

$$\varphi_1(x) = \left(\frac{\alpha}{2\pi^{1/2}}\right)^{1/2} 2\alpha x\, e^{-\alpha^2 x^2/2}$$

where $\alpha = (m\omega/\hbar)^{1/2}$

Ladder operators for harmonic oscillator

$$a_\pm = \frac{1}{\sqrt{2}}\left(\alpha x \mp i\frac{p}{\hbar\alpha}\right)$$

Radial functions for the hydrogen atom $R_{nl}(r)$

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\exp(-r/a_0) \qquad R_{20}(r) = \frac{2}{(2a_0)^{3/2}}[1 - r/(2a_0)]\exp[-r/(2a_0)]$$

$$R_{21}(r) = \frac{r}{24^{1/2}a_0^{5/2}}\exp[-r/(2a_0)]$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q[V(\mathbf{b}) - V(\mathbf{a})]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau = Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$, in which

$q = \int \rho(\mathbf{r})d^3\mathbf{r}$ is the monopole moment

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}d^3\mathbf{r}$ is the dipole moment

$Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right]d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$)

Relative permittivity: $\varepsilon_r = 1 + \chi_e$

Bound charges

$\rho_b = -\nabla \cdot \mathbf{P}$

$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0 \dfrac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

Relative permeability: $\mu_r = 1 + \chi_m$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$ Current densities: $I = \int \mathbf{J} \cdot d\mathbf{A}$, $I = \int \mathbf{K} \cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \int \dfrac{I d\boldsymbol{\ell} \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{encl}$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I \int d\mathbf{a}$

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \dfrac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$

Bound currents

$J_b = \nabla \times \mathbf{M}$

$K_b = \mathbf{M} \times \hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1.	$\nabla \cdot \mathbf{D} = \rho_\mathrm{f}$	Gauss' Law
2.	$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
3.	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
4.	$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f} + \dfrac{\partial \mathbf{D}}{\partial t}$	Ampere's Law with Maxwell's correction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

Wave equations in a conducting medium

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}.$$

Boundary conditions in electrodynamics

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f, \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0$$

$$B_1^\perp - B_2^\perp = 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^\parallel - \frac{1}{\mu_2}\mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$$

$$\int_0^\infty x^n e^{-x} dx = n!$$

$f(x)$	$\int_0^\infty f(x)\,dx$
$e^{-a\,x^2}$	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
$x e^{-a\,x^2}$	$\frac{1}{2\,a}$
$x^2 e^{-a\,x^2}$	$\frac{\sqrt{\pi}}{4\,a^{3/2}}$
$x^3 e^{-a\,x^2}$	$\frac{1}{2\,a^2}$
$x^4 e^{-a\,x^2}$	$\frac{3\sqrt{\pi}}{8\,a^{5/2}}$
$x^5 e^{-a\,x^2}$	$\frac{1}{a^3}$
$x^6 e^{-a\,x^2}$	$\frac{15\sqrt{\pi}}{16\,a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2} dx = \ln\left(x + \sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1} dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2} dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2} dx = \frac{\frac{bx}{x^2+b^2} + \arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$