UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Tuesday, May 24, 2022

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1. A photon with a wavelength of 100 nm hits a 100 g copper pan initially at rest.

a) If the photon is completely absorbed by the copper pan, what is the momentum of the pan?
b) If the copper pan has a work function of 4.5 eV, what is the energy of the photoelectron emitted as a result the absorption of the photon in part (a) ?
c) Assuming that the photoelectron is now moving exactly in the opposite direction to the incoming photon in part (a), what is the momentum of the pan relative to its initial 'rest' condition?

A2. The parity operator $\hat{P}$ in 1 dimension is defined by $\hat{P}\psi(x) = \psi(-x)$

a. Is $\hat{P}$ Hermitian?
b. Find the eigenvalues of $\hat{P}$.
c. Is the plane wave exp(*ikx*) an eigenstate of $\hat{P}$?

A3. We use an electron gun to produce electrons with an energy of 50 eV.

a) What is the de Broglie wavelength of these electrons?
b) If the electrons are monoenergetic but not coherent and are fired through a slit that is 1 nm wide, estimate the width of the electron beam 1 m away from the slit.
c) Using the values above, but now assume the electron beam is coherent. The electrons goe through a double slit aperture, with the slits separated by 3 nm. What is the scattering angle to the FOURTH peak (with the angle defined by the deviation from unperturbed direction of propagation)?

(Note: the center is n=0, then is n=1 (first order diffraction peak) and then n=2 (2nd order diffraction peak) and so on).

A4. (a) Prove that if A is a Hermitian operator, then the expectation value of A^2 is always positive. What can be said about this expectation value if A is antihermitian, that is $A^\dagger = -A$?

(b) Prove that if A is Hermitian, then e^{iA} is unitary.

<u>**Quantum Mechanics Group B** - Answer only two Group B questions</u>

B1. Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by $|\alpha\rangle = i|1\rangle + 2|2\rangle + i|3\rangle$ and $|\beta\rangle = i|1\rangle + 2|3\rangle$.

a. Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$.
b. Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$.
c. Find all nine matrix elements of the operator $\hat{A} = |\alpha\rangle\langle\beta|$ and construct its matrix **A** in the basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Is **A** hermitian?

B2. Find the eigenvalues of the component of the electron spin in the direction of an arbitrary unit vector $\hat{\boldsymbol{n}}$

B3. An electron in a hydrogen atom is in the stationary state

$$\psi_{2,1,-1}(r,\theta,\phi) = Nre^{-r/2a_0}Y_{1,-1}(\theta,\phi)$$

(a) Find the normalization constant *N*. Check that it has the correct unit.
(b) What is the probability density for finding the electron at $r = a_0$, $\theta = 45°$, and $\phi = 60°$?
(c) What is the probability of finding the electron in a thin sperical shell of thickness dr at $r = 2a_0$?
(d) If L^2 is measured, what outcomes can be found and with what probabilities?
(e) If, instead, L_z is measured, what outcomes can be found and with what probabilities?

B4. For an electron (spin 1/2, mass *m*, charge –*e* and *g*-factor *g*=2):

(a) Find the eigenvalues and eigenstates of S_x.

(b) Suppose the electron is in the state corresponding to the larger eigenvalue. Find expectation values of S_z, S_y, S_z^2, and S_y^2 for this state.

(c) Using the results obtained in (b), find $<S_x^2+S_y^2+S_z^2>$.

(d) Now place the electron in a magnetic field *B* directed along the *x* axis. Write the Hamiltonian. Consider the following statement: The solutions found in part (a) are eigenstates of the Hamiltonian. Is this statement correct? If yes, find the corresponding energy eigenvalues. If no, find the expectation values of *H* for these states.

Electrodynamics Group A - Answer only two Group A questions

A1. The electrostatic field produced by a charge density is given by

$$\mathbf{E} = E_0 \frac{y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r}.$$

Find the total charge Q contained in a sphere of radius a centered at the origin.

A2. An infinite plane of a uniform charge σ_0 per unit area is placed at distance $z = h$ above the surface of a half-space grounded metal.

(a) Find the potential and the electric field in all space.

(b) Find the induced surface charge on a metal surface.

A3. An elastic circular conducting loop is expanding at a constant rate so that its radius is given by $r=r_0+vt$. The loop is in a region of constant magnetic field B perpendicular to the loop.

(a) What is the emf generated in the expanding loop? Neglect possible effect of self inductance.
(b) Show the direction of the induced current relative to the direction of the magnetic field

A4. (a) Are the following possible specifications of a B-field? (There is no displacement current)

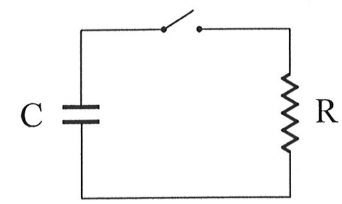
$B_x = a(x+y)$, $B_y = -a(y+z)/2$, $B_z = -a(x+z)/2$.

(b) If yes, find the current density which gives rise to it.

(c) Is there any time-dependent charge density associated with this field?

<u>Electrodynamics Group B</u> - <u>Answer only two Group B questions</u>

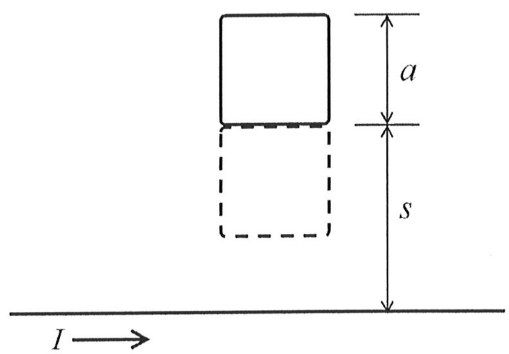
B1. A simple circuit consists of a capacitor C carrying charge Q and a resistor R as shown. At time $t = 0$, the switch is closed and the capacitor is discharging. Find current $I(t)$ as a function of time t and show that the electric energy originally stored on the capacitor is fully dissipated in the resistor R at $t \to \infty$.



B2. A square loop, side a, resistance R, lies at a distance s ($s>a$) from an infinite straight wire that carries current I. The loop performs half a revolution about its bottom side arriving into the position indicated by the dashed square.

(a) Indicate the direction of the induced current in the loop just after it starts the half revolution and just before it reaches its final position (the dashed square).

(b) What total charge passes a given point in the loop?



B3. A dielectric sphere of radius R with a permittivity ε has a free charge Q uniformly distributed over the volume. The sphere is surrounded by empty space.

a) Find the electric field **E** and the electric displacement **D** inside and outside the sphere.

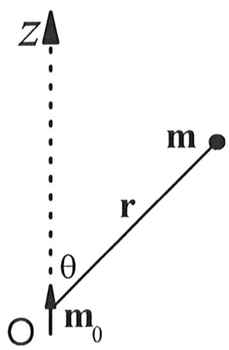
b) Find the polarization **P** and the volume and surface polarization charge densities.

c) Show that the total polarization charge is zero.

B4. A magnetic dipole moment with fixed magnitude and direction, $\mathbf{m}_0 = m_0\hat{\mathbf{z}}$, is held at the origin of coordinates. Another magnetic moment **m** is held fixed at an arbitrary point **r**, but its orientation is allowed to change freely. See the figure below.

(a) Find the equilibrium orientation of the dipole moment **m** (given by the unit vector $\hat{\mathbf{m}}$) that corresponds to the minimum of magnetostatic energy. Express it in terms of $\hat{\mathbf{r}}$, $\hat{\mathbf{z}}$, and θ (the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$).

(b) Draw an arrow indicating the equilibrium orientation of the magnetic moment **m** on the figure below, assuming that the angle between **r** and $\mathbf{m}_0$ is $\theta = 45°$.



Physical constants

speed of light................ $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π... $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant ..... $k_B = 1.381\times10^{-23}$ J/K

elementary charge .... $e = 1.602\times10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability... $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant......... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022\times10^{23}$ mol^{-1}

fine structure constant $\alpha = ke^2/(\hbar c)$

electrostatic const...... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times10^{-31}$ kg

electron rest energy .. 511.0 keV

Compton wavelength . $\lambda_C = h/m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673\times10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2/ke^2m_{el} = 0.5292$ Å

1 hartree (= 2 Ry) $E_h = \hbar^2/m_{el}a_0^{\,2} = 27.21$ eV

gravitational constant $G = 6.674\times10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

1 Ry =13.6 eV

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$

$$\sin\alpha\sin\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$\cos(ix)=\cosh(x)$
$\sin(ix)=i\sinh(x)$

For small x:

$$\sin x\approx x-\tfrac{1}{6}x^3$$
$$\cos x\approx 1-\tfrac{1}{2}x^2$$
$$\tan x\approx x+\tfrac{1}{3}x^3$$

QUANTUM MECHANICS

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x,L_y]=i\hbar L_z \quad \textit{et cycl.}$

Ladder operators:
$$L_+|\ell,m\rangle=\hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\ell,m+1\rangle$$
$$L_-|\ell,m\rangle=\hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\ell,m-1\rangle$$

$$H_{\text{mag}}=-\gamma\,\mathbf{S}\cdot\mathbf{B}$$

Pauli matrices: $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z)=\frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z)=\sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z)=\mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z)=\sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z)=\mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z)=\mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm2ixy}{r^2}$

Radial functions for the hydrogen atom $R_{nl}(r)$

$$R_{10}(r)=\frac{2}{a_0^{3/2}}\exp(-r/a_0) \qquad R_{20}(r)=\frac{2}{(2a_0)^{3/2}}[1-r/(2a_0)]\exp[-r/(2a_0)]$$

$$R_{21}(r)=\frac{r}{24^{1/2}a_0^{5/2}}\exp[-r/(2a_0)]$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da=\frac{q_{\text{encl}}}{\varepsilon_0} \qquad \mathbf{E}=-\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{E}\cdot d\boldsymbol{\ell}=V(\mathbf{r}_1)-V(\mathbf{r}_2) \qquad V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Work done $W=-\int_{\mathbf{a}}^{\mathbf{b}}q\mathbf{E}\cdot d\boldsymbol{\ell}=q[V(\mathbf{b})-V(\mathbf{a})]$ Energy stored in elec. field: $W=\frac{1}{2}\varepsilon_0\int_V E^2d\tau=Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r})=\frac{q}{4\pi\varepsilon_0 r}+\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}+\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij}Q_{ij}\frac{x_ix_j}{r^5}+\ldots$, in which

$q=\int\rho(\mathbf{r})d^3\mathbf{r}$ is the monopole moment

$\mathbf{p}=\int\rho(\mathbf{r})\mathbf{r}d^3\mathbf{r}$ is the dipole moment

$Q_{ij}=\int\rho(\mathbf{r})\left[3r_ir_j-r^2\delta_{ij}\right]d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1=x,\ r_2=y,\ r_3=z$)

Relative permittivity: $\varepsilon_r = 1 + \chi_e$

Bound charges

D=ε₀E + P

$\rho_b = -\nabla \cdot \mathbf{P}$

$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0 \dfrac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

Relative permeability: $\mu_r = 1 + \chi_m$

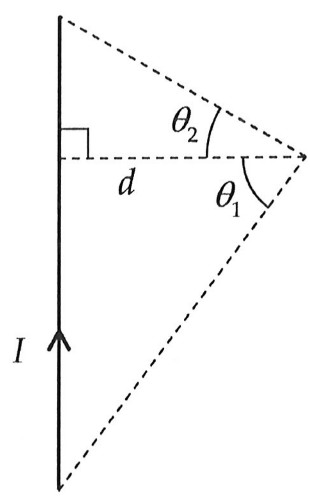
Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

Current densities: $I = \int \mathbf{J} \cdot d\mathbf{A}$, $I = \int \mathbf{K} \cdot d\ell$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \int \dfrac{Id\ell \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\ell = \mu_0 I_{\text{encl}}$



$$B = \frac{\mu_0 I}{4\pi d}(\sin\theta_1 + \sin\theta_2)$$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \dfrac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$

Bound currents

$J_\mathrm{b} = \nabla \times \mathbf{M}$
$K_\mathrm{b} = \mathbf{M} \times \hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Continuity equation: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_\mathrm{f}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f} + \dfrac{\partial \mathbf{D}}{\partial t}$ Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$$

$$\int_0^\infty x^n e^{-x} dx = n!$$

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\frac{1}{2a}$
$x^2e^{-ax^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2} dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2} dx = \ln\left(x + \sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1} dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2} dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2} dx = \frac{\frac{bx}{x^2+b^2} + \arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$