UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 13, 2021

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1. Light of wavelength 400 nm is incident on a metallic surface. If the stopping potential for the photoelectric effect is 1.10 eV, find

(a) The maximum energy of the emitted electrons,
(b) The work function,
(c) The cutoff wavelength

A2. Operator *R* is defined by $R\psi(x) = \mathrm{Re}[\psi(x)]$. Is *R* Hermitian?

A3. In a two-dimensional Hilbert space spanned by the orthonormal kets $|\uparrow\rangle$ and $|\downarrow\rangle$, the operator *F* is defined by

$$F|\uparrow\rangle = |\downarrow\rangle$$
$$F|\downarrow\rangle = |\uparrow\rangle$$

a. Is *F* Hermitian?
b. Find the eigenkets and eigenvalues of *F*.

A4. Positronium is a system consisting of an electron and its anti-particle, positron.

(a) Calculate the energy of positronium in the state with the principal quantum number *n*=2.
(b) Suppose positronium performs transition from the *n*=2 state to the *n*=3 state by absorbing a photon. What is the photon's wavelength?

<u>**Quantum Mechanics Group B** - Answer only two Group B questions</u>

B1. An X-ray photon undergoes Compton scattering. The maximum possible energy which can be transferred to electron is 50 keV.

(a) What is the wavelength and energy of the incident photon?
(b) Suppose the same photon is scattered by 60° How large is the energy transfer in this case? What is the wavelength of the scattered photon?

B2. The wavefunction of a particle moving in one dimension is given by

$$\psi(x) = \begin{cases} 0 & x < -b/2 \\ C & -b/2 < x < +b/2 \\ 0 & x > +b/2 \end{cases}$$

where *C* is a real-valued, positive constant.

a. Normalize the wavefunction.
b. Find $\varphi(k)$, the wavefunction in *k*-space ($k = p/\hbar$).
c. Estimate the widths Δx and Δp, and show that they agree with the Heisenberg's uncertainty principle.

B3. An electron in a hydrogen atom is in the normalized stationary state $\psi_{21\text{-}1}(\mathbf{r})$ (n=2,l=1,m=-1).

a. Calculate the probability that the electron's polar angular coordinate θ is 60 degrees or less. (Hint: for the θ integration use the substitution $u=\cos\theta$).
b. Calculate the probability that the electron's radial coordinate r is greater than a_0 where a_0 is the Bohr radius.
c. What is the expectation value of r?
d. What is the most probable value of r?

B4. A particle of mass *m* is placed in an infinite potential well with the potential energy

$$V(x)=0 \text{ for } -L/2<x<L/2, \quad V(x)=\infty \text{ otherwise}$$

a. Find the normalized wavefunctions of the ground state ψ_1 and the first excited state ψ_2. Make sure you choose the appropriate phase of the wave functions ψ_1 and ψ_2 such that $\psi_1(x=0)>0$ and $\psi_2(x=L/4)>0$.

Suppose at t=0 the wavefunction of the particle is given by

$$\Psi=C(2\psi_1+\psi_2)$$

b. Find the normalization constant *C.*
c. Find the expectation value of the Hamiltonian *H* and expectation value of *x* at *t*=0

d. Find the expectation values of H and x at t>0.

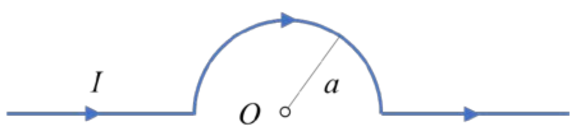
You can use the integral $$\int_{-\pi/2}^{\pi/2} x\cos(x)\sin(2x)dx = 8/9.$$

Electrodynamics Group A - Answer only two Group A questions

A1. A particle of charge q is moved from infinity into the center of a hollow conducting spherical shell of inner radius a and thickness t, through a tiny hole in the shell. How much work is required?

A2. A slab of homogeneous dielectric material of dielectric permittivity ε and thickness d is infinite in the z=0 plane. It is placed in an external field $\mathbf{E}_0 = E_0\mathbf{z}$, where E_0 is a constant. There are no free charges in the slab. Using the electrostatic boundary conditions, find the electric field and induced polarization charge density σ_P on top and bottom surfaces of the slab. Find the electric field $\mathbf{E}_P$ which is produced by the polarization charges and show that $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_P$.

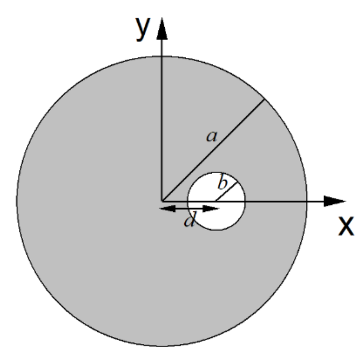
A3. An infinitely long wire carries a current I = 1 A. It is bent so to have a semi-circular detour around the origin with radius a = 1 cm, as shown in the figure below. Calculate the magnetic field at the origin.



A4. A long solenoid with radius a and n turns per unit length carries a current which is growing with time as $I(t)=bt$ where b is a constant. Find the electric field (magnitude and direction) at a distance r from the axis, both inside and outside the solenoid.

<u>Electrodynamics Group B</u> <u>- Answer only two Group B questions</u>

B1. Consider an infinite cylindrical wire oriented along the *z* direction with radius *a*. This wire has an infinite cylindrical cavity parallel to the wire with radius *b*, but displaced from the axis by a distance *d* along the *x* direction (see the cross-section of the wire in the figure below). This wire carries a total current *I* uniformly distributed throughout its cross-section flowing along the +*z* direction. Using Ampere's law and the superposition principle find the magnetic field inside the cavity.



B2. A grounded spherical metal shell of radius R is filled with a space charge of uniform charge density ρ. Find the electric field, the electric potential, and the electrostatic energy of the system.

B3. A solid spherical conductor of a uniform conductivity σ has a uniform volume charge density ρ_0 at time $t = 0$.

a. Find the electric field and the electric current density in the conductor as functions of time . *t*
b. Obtain the field and the current density at $t \to \infty$ and explain the physical meaning of the result

Hint: use Gauss' law, the continuity equation, and the relation between the electric field and the current density, $\mathbf{J} = \sigma\mathbf{E}$.

B4. Consider a plane linearly polarized monochromatic wave of electric field amplitude E_0, frequency ω traveling in the direction from the origin to the point (1,1,1), with polarization parallel to the *xz* plane.

a. Find the Cartesian components of the wavevector **k** and the unit polarization vector **n.**
b. Find the electric and magnetic fields as functions of position **r** and time *t.*
c. Find the Poynting vector as a function of **r** and *t*.
d. Find the energy density as a function of **r** and *t*.

Physical constants

speed of light .... $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant .. $k_B = 1.381\times10^{-23}$ J/K

elementary charge ..... $e = 1.602\times10^{-19}$ C

electric permittivity ... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant...... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022\times10^{23}$ mol^{-1}

fine structure constant $\alpha = ke^2/(\hbar c)$

electrostatic const.... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times10^{-31}$ kg

electron rest energy 511.0 keV

Compton wavelength $\lambda_C = h/m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673\times10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2/ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 Ry) $E_h = \hbar^2/m_{el}a_0^2 = 27.21$ eV

gravitational constant $G = 6.674\times10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

1 Ry =13.6 eV

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$\cos(ix)=\cosh(x)$
$\sin(ix)=i\sinh(x)$

For small x:

$\sin x \approx x - \frac{1}{6}x^3$

$\cos x \approx 1 - \frac{1}{2}x^2$

$\tan x \approx x + \frac{1}{3}x^3$

QUANTUM MECHANICS

$$[AB,C] = A[B,C] + [A,C]B$$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_+ |\ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\ell, m+1\rangle$$
$$L_- |\ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\ell, m-1\rangle$$

Infinite potential well with $V(x)=0$ for $0<x<L$, $V(x)=\infty$ otherwise:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L}$$

Compton formula

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta)$$

Spherical harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

Radial functions for the hydrogen atom $R_{nl}(r)$

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\exp(-r/a_0) \qquad R_{20}(r) = \frac{2}{(2a_0)^{3/2}}[1 - r/(2a_0)]\exp[-r/(2a_0)]$$

$$R_{21}(r) = \frac{r}{24^{1/2}a_0^{5/2}}\exp[-r/(2a_0)]$$

ELECTROSTATICS

$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0}$ $\quad$ $\mathbf{E} = -\boldsymbol{\nabla}V$ $\quad$ $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$ $\quad$ $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ $\quad$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$, in which

$q = \int \rho(\mathbf{r})d^3\mathbf{r}$ is the monopole moment

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}d^3\mathbf{r}$ is the dipole moment

$Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right]d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$)

Relative permittivity: $\varepsilon_{\text{r}} = 1 + \chi_{\text{e}}$

Bound charges

$\rho_{\text{b}} = -\boldsymbol{\nabla}\cdot\mathbf{P}$

$\sigma_{\text{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0 \frac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \frac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

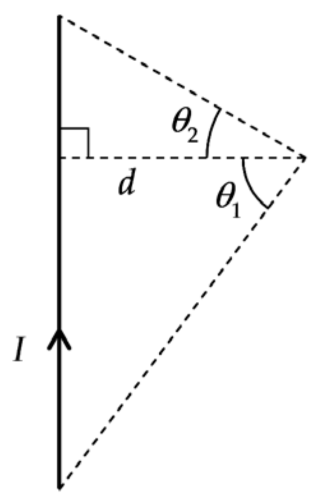
Relative permeability: $\mu_{\text{r}} = 1 + \chi_{\text{m}}$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I d\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl}}$



$$B = \frac{\mu_0 I}{4\pi d}\left(\sin\theta_1 + \sin\theta_2\right)$$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}$

Bound currents

$J_b = \nabla\times\mathbf{M}$

$K_b = \mathbf{M}\times\hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t}$ Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\frac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Plane electromagnetic wave

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} \qquad u = \frac{|\mathbf{S}|}{c}$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$$

$$\int_0^\infty x^n e^{-x} dx = n!$$

$f(x)$	$\int_0^\infty f(x)\,dx$
$e^{-a x^2}$	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
$x e^{-a x^2}$	$\frac{1}{2a}$
$x^2 e^{-a x^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-a x^2}$	$\frac{1}{2a^2}$
$x^4 e^{-a x^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-a x^2}$	$\frac{1}{a^3}$
$x^6 e^{-a x^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2} dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2} dx = \ln\left(x + \sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1} dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2} dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2} dx = \frac{\frac{bx}{x^2+b^2} + \arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$