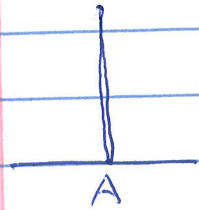
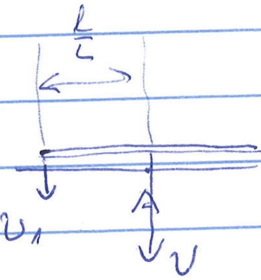
CM

A1.



6 

horizontal position of cm is the same

Conservation of energy

8 $$mg\frac{l}{2} = \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

where ω is the angular velocity about the cm,

$$I = \frac{ml^2}{12}$$

Since the vertical velocity of the left end is 0

5 $$v_1 = v - \omega\frac{l}{2} = 0$$

$$\omega = \frac{2v}{l}$$

2 $$mg\frac{l}{2} = \frac{mv^2}{2} + \frac{1}{2}\frac{ml^2}{12}\frac{4v^2}{l^2}$$

2 $$gl = v^2 + \frac{1}{3}v^2$$

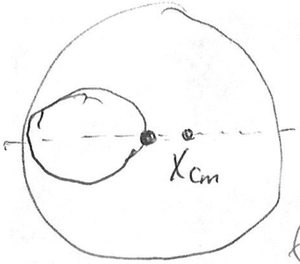
2 $$v = \sqrt{\frac{3}{4}gl} = \frac{1}{2}\sqrt{3gl}$$

$$\omega = \sqrt{\frac{3g}{l}}$$

$$\cancel{T = \frac{m\omega^2 l^2}{8} + \frac{m}{24}\omega^2 l^2 = \frac{1}{6}m\omega^2 l^2}$$

A2

(a) CM



6

For the whole sphere $X_{cm} = 0$

$$0 = \frac{\frac{7}{8} m_0 \cdot X_{cm} - \frac{1}{8} m_0 \frac{a}{2}}{m_0}$$

2 $\quad 7 X_{cm} = \frac{a}{2} \qquad X_{cm} = \frac{a}{14}$

6 (b) $I = \frac{2}{5} m_0 a^2 - \frac{2}{5} \frac{m_0}{8} \left(\frac{a}{2}\right)^2$

where m_0 is the mass of the whole sphere

6 $\quad m = \frac{7}{8} m_0 \qquad m_0 = \frac{8}{7} m$

5 $\quad I = \frac{2}{5} m_0 a^2 \left(1 - \frac{1}{32}\right) = \frac{31}{80} m_0 a^2 = \frac{31}{70} m a^2$

A3 CM

Conservation of energy

10 $$\frac{m v_0^2}{2} = \frac{m v^2}{2} + \frac{m \omega_0^2 x^2}{2}$$

(a) Solve for v

(b) 3 $$v = (v_0^2 - \omega_0^2 x^2)^{1/2}$$

if $T = V$

8 $$\frac{m v_0^2}{2} = m \omega_0^2 x_1^2$$

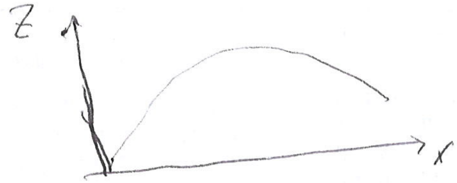
$$x_1 = \frac{1}{\sqrt{2}} \frac{v_0}{\omega_0}$$

oscillation amplitude $x_0 = \frac{v_0}{\omega_0}$

4

therefore $\frac{x_1}{x_0} = \frac{1}{\sqrt{2}}$

CM A4



4 $$H = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + mgz$$

2 $$\frac{\partial H}{\partial p_x} = \dot{x} \longrightarrow \frac{p_x}{m} = \dot{x} \quad (1)$$

2 $$\frac{\partial H}{\partial p_z} = \dot{z} \longrightarrow \frac{p_z}{m} = \dot{z} \quad (2)$$

2 $$\frac{\partial H}{\partial x} = -\dot{p}_x \longrightarrow 0 = -\dot{p}_x \quad (3)$$

2 $$\frac{\partial H}{\partial z} = -\dot{p}_z \longrightarrow mg = -\dot{p}_z \quad (4)$$

From (1) and (3) $p_x = const$, $\dot{x} = const$

2 From (4) $p_z = p_{z0} - mgt$

2 From (2) $\dot{z} = \frac{p_{z0}}{m} - gt$

This eq. is obtained from the 2nd Newton's law

5 $$m\ddot{z} = -mg \rightarrow \dot{z} = (\dot{z})_0 - gt$$

similar to the x direction

2 $$m\ddot{x} = 0 \quad \dot{x} = const \quad p_x = const$$

CM

B1

2nd Newton law for rot. motion

5 $$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} \qquad \text{where } \vec{r} = r\vec{e}_r$$

$$\vec{F} = \vec{e}_r f(r) - \lambda\dot{\vec{r}} = \vec{e}_r f(r) - \lambda(\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)$$

5 therefore $\vec{r}\times\vec{F} = -\lambda r^2\dot{\theta}\,\vec{e}_r\times\vec{e}_\theta = -\lambda r^2\dot{\theta}\,\vec{e}_z$

where $\vec{e}_z$ is unit vector perpendicular to the plane of motion

(a) $$\frac{d\vec{L}}{dt} = -\lambda r^2\dot{\theta}\,\vec{e}_z$$

4 in polar coordinates $\vec{L} = m\,\vec{r}\times\vec{v} = mr^2\dot{\theta}\,\vec{e}_z$

therefore equation of motion is reduced to

3 $$\frac{d\vec{L}}{dt} = -\frac{\lambda}{m}\vec{L}$$

(b) 3 Solution $\vec{L} = L\vec{e}_z$, $\longrightarrow$ $\frac{dL}{dt} = -\frac{\lambda}{m}L$

5 $L = L_0 e^{-\frac{\lambda}{m}t}$

CM

B2

$$v_i = v_0 \cos\frac{\theta_0}{2} \quad (1)$$

3 $$x = v_i t \cos\theta_0 \qquad y = v_i t \sin\theta_0 - \frac{gt^2}{2}$$

the time of flight is from $y=0$

2 $$t = \frac{2 v_i \sin\theta_0}{g}$$

and the range is

5 $$X_{max} = v_i \frac{2 v_i \sin\theta_0}{g}\cos\theta_0 = \frac{2 v_i^2}{g}\sin\theta_0\cos\theta_0$$

substitute here (1)

2 $$X_{max} = \frac{2 v_0^2}{g}\cos^2\frac{\theta_0}{2}\sin\theta_0\cos\theta_0 = \frac{v_0^2}{g}(1+\cos\theta_0)\cos\theta_0\sin\theta_0$$

with $u = \cos\theta_0$, we have to maximise the function

3 $$f(u) = (1+u)u\sqrt{1-u^2}$$

5 $$\frac{df}{du} = u\sqrt{1-u^2} + (1+u)\sqrt{1-u^2} - (1+u)u\frac{u}{\sqrt{1-u^2}} = 0 \qquad u \neq 1$$

$$(1+2u)\overset{(1-u)(1+u)}{\cancel{(1-u^2)}} - u^2(1+u) = 0 \qquad u \neq -1$$

4 $$\cancel{3u^3}\ (1+2u)(1-u) - u^2 = 0$$

$$-3u^2 + u + 1 = 0$$

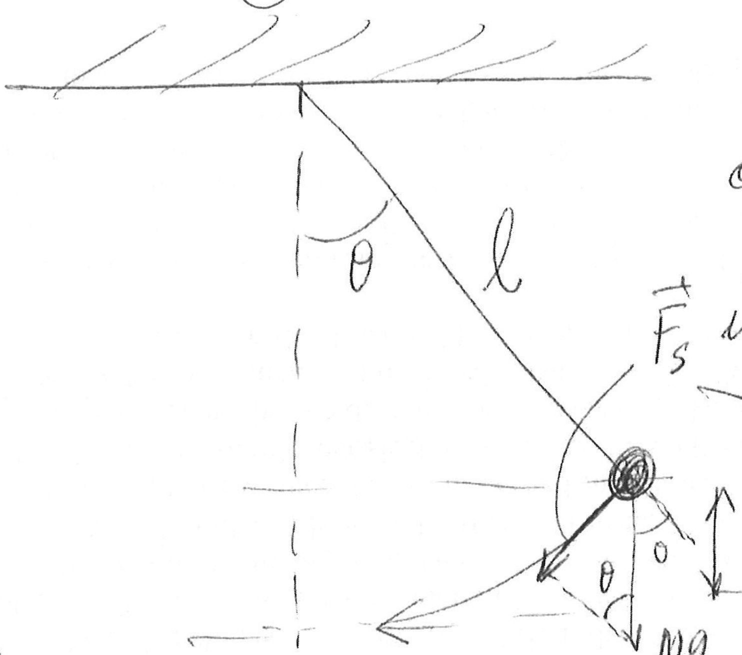
$$u = \frac{1 \pm \sqrt{1+12}}{6} = \frac{1 \pm \sqrt{13}}{6}$$

1 phys. solution $\cos\theta_0 = \frac{1+\sqrt{13}}{6} = 0.7676$, $\sin\theta_0 = 0.6409$

$$\theta_0 = 40^\circ \qquad X_{max} = 0.8696\frac{v_0^2}{g}$$

5 CM B3

pb # 4 (hard)



a) not drag force

$\vec{F}_s$ is a restoring force $= mg\sin\theta$

Lagrangian

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (*)$$

$$v = l\dot\theta \Rightarrow T = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}ml^2\dot\theta^2$$

$$V = mgl(1-\cos\theta) \simeq mgl\left[1-\left(1-\frac{\theta^2}{2!}+\cdots\right)\right] \simeq \frac{1}{2}mgl\theta^2$$

6 $$L = \frac{1}{2}ml^2\dot\theta^2 - \frac{1}{2}mgl\theta^2$$

$$(*) \Leftrightarrow ml^2\ddot\theta + mgl\theta = 0$$

5 $$\Leftrightarrow \boxed{\ddot\theta + \frac{g}{l}\theta = 0}$$

$$\boxed{\begin{aligned}\theta(t) &= \theta_0\cos(\omega_0 t + \beta)\\ \omega_0^2 &= g/l\end{aligned}}$$

B3, p. 2

(6) b) linear drag force: $F_{drag} = -c\dot{s}$,

$s = l\theta$

Newton 2nd law: $\vec{F} = m\vec{a}$ (for translational motion)

4 $\Rightarrow m\ddot{s} = -mg\sin\theta - c\dot{s}$

$\Rightarrow \ddot{s} + \frac{c}{m}\dot{s} + g\sin\theta = 0$

Small angle: $\sin\theta \simeq \theta = \frac{s}{l}$

2 $\Rightarrow \boxed{\ddot{s} + \frac{c}{m}\dot{s} + \frac{g}{l}s = 0 \quad \text{or} \quad \ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{l}\theta = 0}$

5 Solution $\theta(t) = \theta_0 e^{-\gamma t}\sin(\omega_d t + \phi_0)$

where $\gamma = \frac{c}{2m}$, $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$

3 if $\omega_0^2 > \gamma^2$, i.e., $\frac{g}{l} > \frac{c^2}{4m^2}$

alternate

Newton's law for rotational motion

$\tau_{net} = I\ddot{\theta} \Leftrightarrow -mgl\sin\theta - f_{drag}\,l = I\ddot{\theta}$

$\Leftrightarrow -mgl\sin\theta - cl\dot{s} = ml^2\ddot{\theta}$

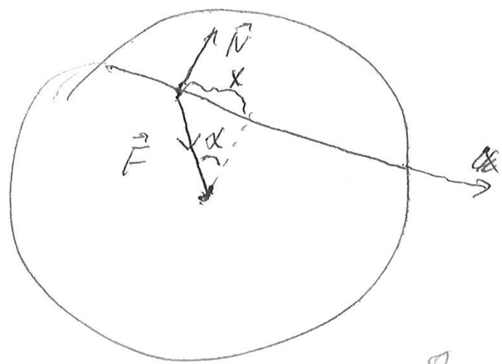
$s = l\theta$

$\dot{s} = l\dot{\theta}$

$\Rightarrow \boxed{\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0}$

$\sin\theta \simeq \theta$ for small θ

A3 B4 CM



6 $F\cos\alpha = N$

$m\ddot{x} = F\sin\alpha$

x is displacement relative to the center of the tunnel

gravity force inside the sphere from Gauss' law for the gravity

8 $\cancel{G\frac{mM}{r^2}} F \cdot 4\pi r^2 = \frac{4}{3}\pi r^3 \rho\, G \cdot 4\pi m$

M_{enc}

$F = \frac{4}{3}\pi r G \rho m$

$m\ddot{x} = -\frac{4}{3}\pi G \rho m r \sin\alpha$ (force is opposite to displacement)

2 $r\sin\alpha = x$

$\ddot{x} = -\frac{4}{3}\pi G \rho x$

harmonic motion:

4 $\ddot{x} = -\omega^2 x$

$\omega^2 = \frac{4}{3}\pi G \rho = 1.33 \cdot 3.14 \cdot 6.67\times10^{-11} \cdot 5.51\times10^3$

$= 1.54\times10^{-6}\ s^{-2}$

5 $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1.54\times10^{-6}}} = 5.06\times10^3 s = 1.41 h$

Preliminary Thermal – August 2025

Easy Problems:

A1. A diatomic gas ($C_v = 2.5\ nR$) expands adiabatically to a volume 1.35 times larger than the initial volume. The initial temperature is 18 °C. Find the final temperature.

Solution: (25 pt)

$C_v dT = -P dV = -nRT dV/V$ (5 pt)

$C_v dT/T = -nR dV/V$, so C_v d(ln$T$) = $-nR$ d(ln$V$) (5 pt)

$d(\ln T^{C_v} + \ln V^{nR}) = 0$

$\ln T^{C_v} + \ln V^{nR} = \text{constant}$

$T^{C_v} V^{nR} = \text{constant}$ (5 pt)

Take the ratio between the initial and final states

$(T_2/T_1)^{C_v} (V_2/V_1)^{nR} = 1$ (5 pt)

Or $(T_2/T_1)^{C_v/nR} (V_2/V_1) = 1$

$V_2/V_1 = 1.35$, so $T_2/T_1 = 0.887$

$T_1 = 18+273.16 = 291.16$ K

$T_2 = 258.22$ K or -14.94 °C (5 pt)

A2. At 0 °C and 1 atm (1atm = 1.01×10^5 Pa) the thermal expansion coefficient for copper is $\alpha=4.85\times10^{-5}$/K, and the isothermal compressibility is $K_T=7.8\times10^{-7}$/atm. If the temperature of the cooper is increased to 10 °C, what's the value of pressure if the volume is kept constant?

Solution: (25 pt)

Using the cyclic rule:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{-1}{\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T}$$ (5 pt)

$$= -\frac{\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T}$$ (5 pt)

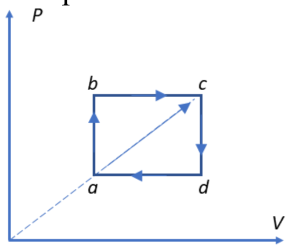
$$= -\frac{\alpha V}{-VK_T} = \frac{\alpha}{K_T}$$ (5 pt)

At constant volume, $\frac{\Delta P}{\Delta T} = \frac{\alpha}{K_T}$,

$$\Delta P = \Delta T\frac{\alpha}{K_T} = 622\ atm$$ (5 pt)

$$P = \Delta P + P_0 = 622 + 1 = 623\ atm$$ (5 pt)

A3. Consider the five processes in the P-V space below. Processes bc and da are isobaric. Processes ab and cd are isochoric. The extension of the process ac passes the origin. Sketch the five processes in the P-T space and the V-T space.

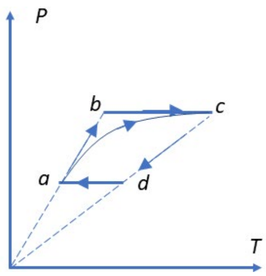


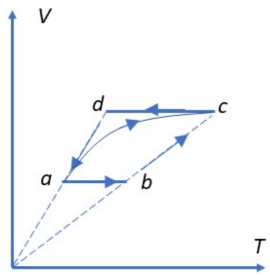
Solution: (25 pt)

$PV = nRT$ (5 pt)

In the P-T space, the extension of processes ab and cd pass the origin (5 pt).

In the V-T space, the extension of processes bc and da pass the origin (5 pt).





(each graph 5 pt)

Thermo (A4)

pb ~~#15~~ : Thermo (Easy)

In the process of expansion at constant pressure p, assuming that the volume increases from V_1 to V_2 and the temperature changes from T_1 to T_2, we have

5 $$\begin{cases} pV_1 = nRT_1 \\ pV_2 = nRT_2 . \end{cases}$$

In this process, the work done by the system on the outside world is

5 $$W = p(V_2 - V_1) = nR\,\overset{(T_2 - T_1)}{\Delta T}$$

and the increase of the internal energy of the system is:

$$\left(C_v = \frac{5R}{2} \right)$$

5 $$\Delta U = C_v \Delta T$$

Therefore:

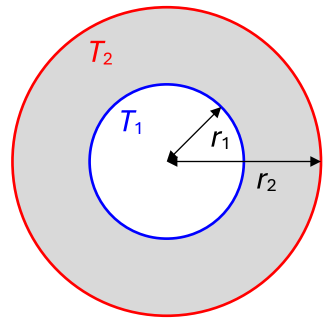
5 $$\frac{W}{Q} = \frac{W}{\Delta U + W} = \frac{nR}{C_v + nR} = \frac{2}{7} .$$

In the process of expansion at constant temperature, the internal energy does not change. Hence

5 $$\frac{W}{Q} = 1 .$$

Hard Problems:

B1. Consider a cylindrical wall with inner radius r_1, outer radius r_2. The temperature for the inner and outer surfaces are T_1 and T_2, respectively. The heat flow is along the radial direction. Find the thermal conductance $\sigma \equiv I/(T_2-T_1)$ of the cylindrical shell, where $I = \lim_{\Delta t\to 0}(\Delta Q/\Delta t)$ is the thermal current, assuming the length of the cylinder is L and thermal conductivity is k.



Solution: (25 pt)

Consider the general Fourier's law:

$I = \lim_{\Delta t\to 0}(\Delta Q/\Delta t) = -kA\left(\frac{\partial T}{\partial r}\right)_t$ (5 pt)

$A = 2\pi rL$, so $\frac{dQ}{dt} = -2\pi rLk\frac{dT}{dr}$. (5 pt)

Along the radius, $Q' = \frac{dQ}{dt}$ is constant. (5 pt)

Hence $\frac{Q'}{r}dr = -2\pi kLdT,\ Q'\ln(r_2/r_1) = -2\pi kL(T_2 - T_1)$ (5 pt)

$\sigma = \frac{Q'}{T_2-T_1} = 2\pi kL/\ln\left(\frac{r_2}{r_1}\right)$ (5 pt)

B2. The speed of sound in air is $C = [(\partial P/\partial\rho)_S]^{0.5}$, where S is entropy, ρ is density. Assume that air can be approximately treated as ideal gas whose molar mass is M.

(a) Derive the relation between γ and C, where $\gamma = C_P/C_V$.

(b) Using the relation C_P-$C_V = nR$, find C_V from the above relation between γ and C.

Solution: (25 pt)

(a) The isentropic process follows PV^{γ} = constant. (5 pt)

(b) Considering that $\rho = M/V$, where V is the molar volume, one has $P\rho^{-\gamma}$ = constant. (5 pt)

(note "-" in the superscript here and below)

Taking differential of both sides, it follows that

$dP\,\rho^{-\gamma} - \gamma P\,\rho^{-\gamma-1}d\rho = 0$ (3 pt)

Hence, $dP/d\rho = \gamma P/\rho$ (3 pt)

Substitute $P = RT/V$, it follows that

$dP/d\rho = \gamma RT/\rho V = \gamma RT/M$ (3 pt)

Since this is an isentropic process, $\gamma RT/M = C^2$,

so $\gamma = C^2M/RT$ (5 pt)

(c) From C_P-$C_V = nR$ and $\gamma = C_P/C_V$, we can find

$C_V = nR/(\gamma\text{-}1)$. (3 pt)

Therefore,

$C_V = nR\,/(C^2M/RT\,\text{-}1)$. (3 pt)

(8) pb ~~#6~~ (B3) Thermo (hard)

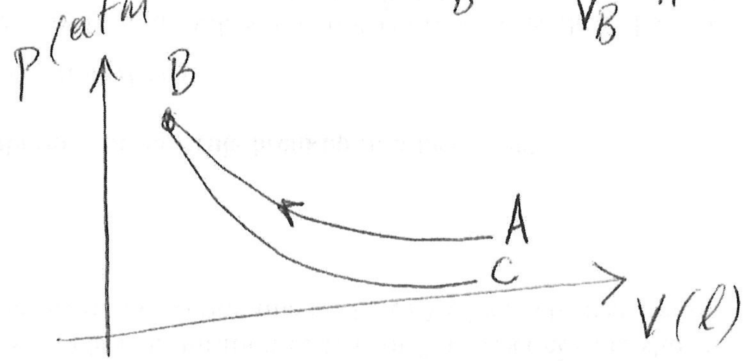
We are given that $V_A = 10\,\ell$ $V_B = 1\,\ell$ $V_C = 10\,\ell$
and $P_A = 1$ atm.

a) $A \longrightarrow B$ process: is an isothermal process, thus

~~b)~~ $$pV = \text{const.} \Rightarrow P_A V_A = P_B V_B$$

4 $$\Rightarrow P_B = \frac{V_A}{V_B} P_A = 10 \text{ atm}$$

3



b) 2 the curve AB for the two kinds of gas are the same.

$B \longrightarrow C$ process: is an adiabatic process, thus

$$pV^\gamma = \text{const.} \Rightarrow P_B V_B^\gamma = P_C V_C^\gamma$$

4 $$\Rightarrow P_C = \left(\frac{V_B}{V_C}\right)^\gamma P_B = 10^{1-\gamma}$$

- monatomic gas: $\gamma = 5/3 \Rightarrow P_C = 10^{-2/3} = 0.215$ atm

4

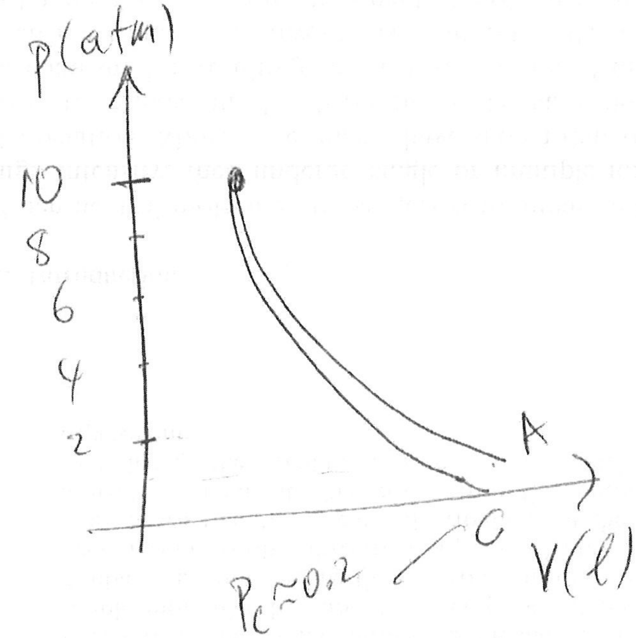
- diatomic gas: $\gamma = 7/5 \Rightarrow P_C = 10^{-2/5} = 0.398$ atm

the curve BC of the monatomic gas is lower than that of

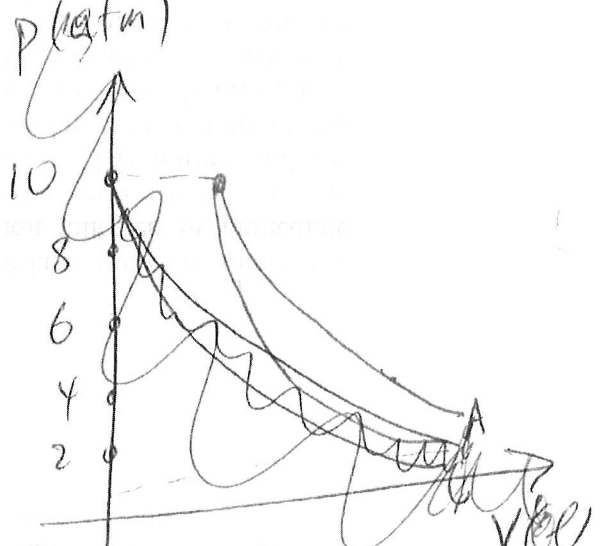
(9) the diatomic gas. (B3), p.2

c) In each case, as the curve AB for compression
3 is higher than the curve BC for expansion, net
work is done on the system.

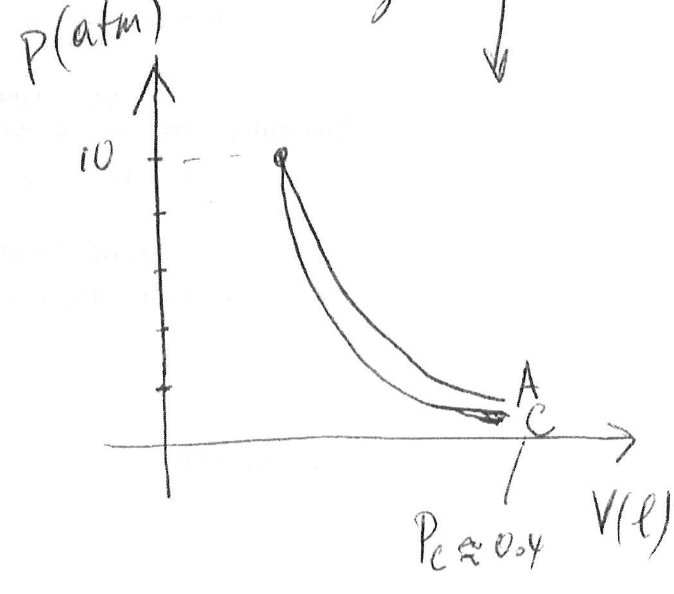
d) As P_C (monatomic gas) < P_C (diatomic gas),
5 the work on the monatomic gas is greater
than on the diatomic gas.



monatomic gas



diatomic gas.



Thermo B4

B4

B1 Let heat capacity be C for each body.
Heat balance (T_f is final T):

$$C(T_f - T_1) + C(T_f - T_2) = 0.$$

5 $$T_f = \frac{T_1 + T_2}{2}.$$

Entropy change:

5 $$\Delta S = \int_{T_1}^{T_f} C \frac{dT}{T} + \int_{T_2}^{T_f} C \frac{dT}{T} =$$

5 $$= C\left(\ln\frac{T_f}{T_1} + \ln\frac{T_f}{T_2}\right) = C \ln\frac{T_f^2}{T_1 T_2} =$$

$$= C \ln\frac{(T_1+T_2)^2}{4T_1T_2} = 2C \ln\frac{T_1+T_2}{2\sqrt{T_1T_2}}$$

10 $\Delta S > 0$ as long as $T_1 \neq T_2$, because the arithmetic mean is larger than the geometric mean, and $\ln x > 0$ for $x > 1$.