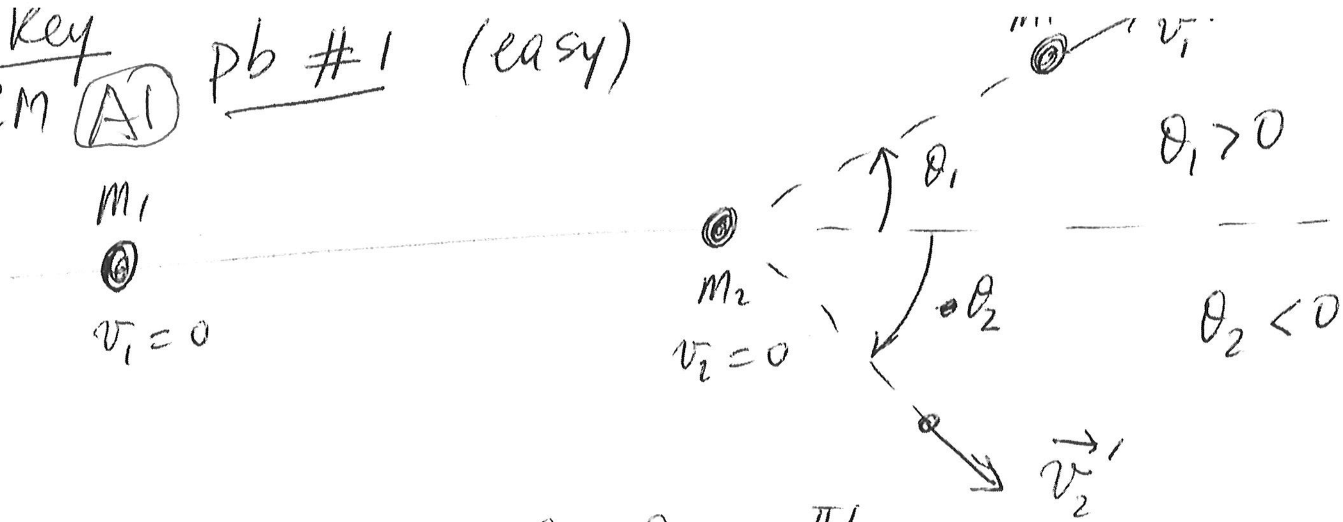
④ Key

CM A1 pb #1 (easy)



Show that if $\theta_1 - \theta_2 = \pi/2$, then the collision is necessarily elastic for identical balls.

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- momentum conservation : $\vec{P}_1 = \vec{P}_1' + \vec{P}_2'$

$$\Rightarrow \quad P_1'^2 + P_2'^2 + 2P_1'P_2' \cos(\theta_1 - \theta_2) = P_1^2 \quad (1)$$

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- If the collision is elastic, then kinetic energy is conserved $K_i = K_f \iff \dfrac{P_1^2}{2m_1} = \dfrac{P_1'^2}{2m_1} + \dfrac{P_2'^2}{2m_2} \quad (2)$

for $m_1 = m_2 \Rightarrow \quad P_1'^2 + P_2'^2 = P_1^2 \quad (3)$

(1) and (3) must be consistent $\iff 2P_1'P_2' \cos(\theta_1 - \theta_2) = 0$

$$\Rightarrow \boxed{\theta_1 - \theta_2 = \pi/2}$$

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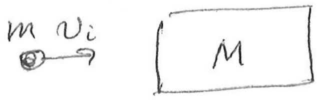
(1) Can also be expressed as

$$P_1^2 = |\mathbb{P}_1'|^2 + 2\,\mathbb{P}_1'\cdot\mathbb{P}_2' + |\mathbb{P}_2'|^2 = |\mathbb{P}_1' + \mathbb{P}_2'|^2$$

to be consistent with (3) we have $\mathbb{P}_1'\cdot\mathbb{P}_2' = 0$ so $\mathbb{P}_1' \perp \mathbb{P}_2'$

Same argument but expressed differently

CM (A2)



conservation of momentum

$$m v_i = (m+M) v$$

$$v = \frac{m v_i}{M+m}$$

$v^2 = 2aS$ where S is the distance travelled

$$a = \frac{F}{m+M} = \mu g$$

$$\left(\frac{m v_i}{m+M}\right)^2 = 2\mu g S$$

$$\mu = \frac{1}{2gS}\left(\frac{m v_i}{m+M}\right)^2 = \frac{1}{\underbrace{19.6}_{2\times 9.8}\times 4}\left(\frac{0.01\times 430}{1.01}\right)^2 = 0.2312$$

If a student takes $g = 10$, then

$$\mu = \frac{1}{80}\left(\frac{430\times 0.01}{1.01}\right)^2 = 0.2266$$

CM A3

Conservation of energy

$$\frac{m v_0^2}{2} - \frac{GmM}{r_0} = \frac{mv^2}{2} - \frac{GmM}{r}$$ 15

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$r = 2.000 \times 10^{12}$ m
$r_0 = 88.00 \times 10^9$ m
$v_0 = 54.47 \times 10^3 \frac{m}{s}$

$$v^2 = (54.47 \times 10^3)^2 + 2 \cdot 1.327 \times 10^{20} \left(\frac{1}{2 \times 10^{12}} - \frac{1}{88 \times 10^9}\right)$$

$$= 2.967 \times 10^9 + 2.654 \times 10^{20} (-10.86 \times 10^{-12})$$ 6

$$= (2.967 - 2.882) \times 10^9 = 0.85 \times 10^8 \frac{m^2}{s^2}$$

$$v = 0.92 \times 10^4 \frac{m}{s} = 9.2 \frac{km}{s}$$

Subtraction of two close numbers gives two-digit accuracy 4

Other ways of computing the answer may not require as much precision to get the answer. Ask if you are unsure.

CM (A4)

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \qquad \vec{v} = \vec{v}_2 - \vec{v}_1$$ 10

Solve these equations for $\vec{v}_1, \vec{v}_2$:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m\vec{v}_{cm}$$
$$\vec{v}_2 - \vec{v}_1 = \vec{v}$$

$$\frac{m_2}{m_1}\vec{v}_2 + \vec{v}_2 = \frac{m}{m_1}\vec{v}_{cm} + \vec{v}$$

$$\frac{m}{m_1}\vec{v}_2 = \frac{m}{m_1}\vec{v}_{cm} + \vec{v} \longrightarrow \vec{v}_2 = \vec{v}_{cm} + \frac{m_1}{m}\vec{v}$$ 8

Similar $\vec{v}_1 = \vec{v}_{cm} - \frac{m_2}{m}\vec{v}$

$(\vec{v}_1 = \vec{v}_2 - \vec{v})$

$$\frac{m_1\vec{v}_1^{\,2}}{2} + \frac{m_2\vec{v}_2^{\,2}}{2} = \frac{m_1}{2}\left(\vec{v}_{cm} - \frac{m_2}{m}\vec{v}\right)^2 + \frac{m_2}{2}\left(\vec{v}_{cm} + \frac{m_1}{m}\vec{v}\right)^2$$

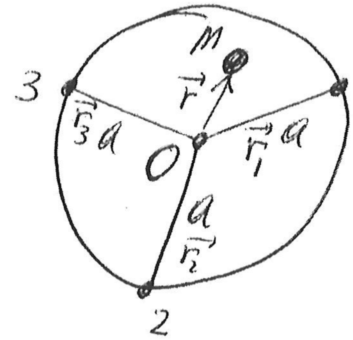
$$= \frac{m_1+m_2}{2}\vec{v}_{cm}^{\,2} + \vec{v}_{cm}\cdot\vec{v}\left(-\frac{m_1m_2}{m} + \frac{m_1m_2}{m}\right) + \vec{v}^{\,2}\left(\frac{m_1m_2^2}{2m^2} + \frac{m_2m_1^2}{2m^2}\right)$$

$$= \frac{m}{2}\vec{v}_{cm}^{\,2} + \frac{1}{2}\frac{m_1m_2(m_1+m_2)}{m^2}\vec{v}^{\,2} = \frac{m}{2}\vec{v}_{cm}^{\,2} + \frac{1}{2}\mu\vec{v}^{\,2}$$ 7

CM Pb # 3 (B1) : Hard

(B1)

a) let $\vec{r}_1, \vec{r}_2, \vec{r}_3$ be the position vectors of the 3 point sources, we have:



$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = \vec{0} \quad (1)$$ since the center O is the center of mass of that system.

The force acting on the particle m is:

$$\vec{F} = -k(\vec{r}-\vec{r}_1) - k(\vec{r}-\vec{r}_2) - k(\vec{r}-\vec{r}_3) = \underline{\underline{-3k\vec{r}}} \quad (2)$$

because we have used Eq. (1).

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b) Newton 2nd law : $\vec{F} = m\ddot{\vec{r}}$ (3)

$$\Longleftrightarrow \boxed{m\ddot{\vec{r}} + 3k\vec{r} = 0} \quad (4)$$

5

which has the general solution

$$\boxed{\vec{r}(t) = \vec{A}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + \vec{B}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)}$$

5

with $\vec{A}$ and $\vec{B}$ being constants to be determined by using the following initial conditions: $\vec{r}(0) = \vec{r}_0$ and $\dot{\vec{r}}(0) = v_0$

(9) One gets: B1, p.2

$$\vec{A} = \vec{r}_0 \quad \text{and} \quad \vec{B}\sqrt{\frac{3k}{m}} = \vec{v}_0$$

$$\Rightarrow \boxed{\vec{A} = \vec{r}_0 \,, \quad \vec{B} = \sqrt{\frac{m}{3k}}\,\vec{v}_0}$$

5

<u>Conclusion</u>:

$$\boxed{\vec{r}(t) = \vec{r}_0 \cos\left(\sqrt{\frac{3k}{m}}\,t\right) + \sqrt{\frac{m}{3k}}\,\vec{v}_0 \sin\left(\sqrt{\frac{3k}{m}}\,t\right)}$$

c) It is obvious that for the trajectory for m to be a circle, one must have:

$$\boxed{\begin{cases} \vec{r}_0 \perp \vec{v}_0 \\ \text{and} \\ r_0 = \sqrt{\frac{m}{3k}}\,v_0 \end{cases}}$$

~~so that~~ 5 (see below)

since

$\vec{v}(t) = -\omega \vec{r}_0 \sin\omega t + \vec{v}_0 \cos\omega t$

the condition $\vec{r} \perp \vec{v}$

leads to $\vec{v}_0 \perp \vec{r}_0$

also $r^2 = r_0^2 \cos^2\omega t + \frac{1}{\omega^2} v_0^2 \sin^2\omega t$

for a circular motion we need equal amplitudes

$\rightarrow r_0 = \frac{v_0}{\omega} = \sqrt{\frac{m}{3k}}\,v_0$

Alternatively, one can express the condition that the particle moves in a circle as

$$|\mathbf{r}(t)| = \text{const} \quad \text{or} \quad |\mathbf{r}(t)|^2 = \text{const} = |\mathbf{r}_0|^2$$

So $|\mathbf{r}(t)|^2 = |\mathbf{r}_0|^2 \cos^2(\omega t) + 2\sqrt{\frac{m}{3k}}\, \mathbf{r}_0 \cdot \mathbf{v}_0 \cos\omega t \sin\omega t$

$$+ \frac{m}{3k} |\mathbf{v}_0|^2 \sin^2\omega t \qquad \left(\omega = \sqrt{\frac{3k}{m}}\right)$$

To make this constant, we need

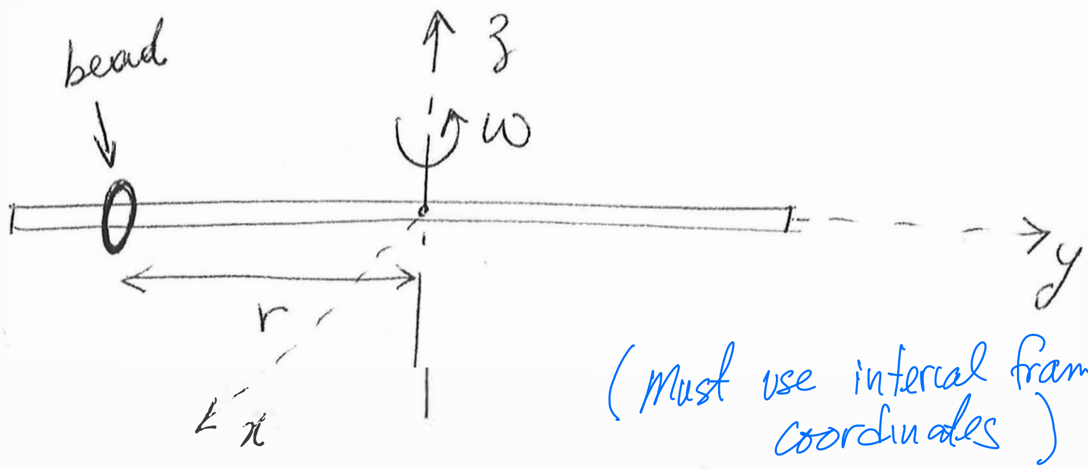
$$\mathbf{r}_0 \cdot \mathbf{v}_0 = 0 \quad \& \quad |\mathbf{r}_0|^2 = \frac{m}{3k} |\mathbf{v}_0|^2$$

So that $|\mathbf{r}(t)|^2 = |\mathbf{r}_0|^2 (\cos^2\omega t + \sin^2\omega t)$

$$= |\mathbf{r}_0|^2$$

(10) CM (B2)

pb # 4 : hard.



(Must use inertial frame coordinates)

a) $r \equiv$ generalized coordinate

$$L = T - V \quad , \quad V = 0$$

transformation equation : $\begin{cases} x = r\cos(\omega t) \\ y = r\sin(\omega t) \end{cases}$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2)$$

b) Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \iff m\ddot{r} - mr\omega^2 = 0$$

$$\Rightarrow \boxed{\ddot{r} - \omega^2 r = 0}$$

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

From $r(0) = r_0$, $\cdot r(0) = 0$ we get $A + B = r_0$, $A - B = 0$, and $A = B = r_0/2$

$$r(t) = \frac{r_0}{2} \cosh \omega t$$

outwards

(II) d) Newton's 2nd law for rotational moti[on]
or Lagrange's equation including multipl[iers]

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net} \iff \frac{d}{dt}(mr^2\omega) = rF$$

where F is acting $\perp$ to the wire and to the rotation axis

$$\Rightarrow 2mr\omega\dot{r} = rF$$

$$\Rightarrow F = 2m\omega\dot{r}$$

but $\dot{r} =$

$$\dot{r} = \frac{r_0\omega}{2}\sinh\omega t$$

$$\Rightarrow \boxed{F = 2m\omega\dot{r} = m\omega^2 r_0 \sinh\omega t}$$

This result can also be obtained from the Coriolis force.

(5) CM (B3) Pb # 2 : ~~easy~~ ~~hard~~ ~~easy~~

a) equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Rightarrow \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0} \quad (1) \qquad \omega_0^2 = \frac{k}{m}, \quad \gamma = \frac{c}{2m}$$

solution of Eq. (1)?

$$x = x_0 e^{rt} \Rightarrow \dot{x} = rx, \quad \ddot{x} = r^2 x \quad (2)$$

$$(2) \text{ in } (1) \Rightarrow r^2 + 2\gamma r + \omega_0^2 = 0$$

$$\Delta = 4\gamma^2 - 4\omega_0^2 = 4(\gamma^2 - \omega_0^2)$$

For underdamped harmonic oscillator, $\Delta < 0$

$\Rightarrow \gamma < \omega_0$, Thus $\Delta = -4(\omega_0^2 - \gamma^2) < 0$

solutions in r are : $\underline{\underline{r_{1,2} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}}}$

$$\omega_d \equiv \sqrt{\omega_0^2 - \gamma^2} \Rightarrow r_{1,2} = -\gamma \pm i\omega_d$$

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solutions of (1) are thus : $x = x_0^+ e^{r_1 t} + x_0^- e^{r_2 t}$

we can show that ~~$x = x_0 e^{\gamma t}\omega($~~

$$\boxed{x = e^{-\gamma t}(A\cos(\omega_d t + \phi_0))} \quad (3)$$

or $x = e^{-\gamma t} A \ldots$

(v) or $x = e^{-\gamma t} A \sin(\omega_d t + \phi)$, $\theta = \omega_d t$ B3, p.2

b) $\dot{x} = -A e^{-\gamma t}(\gamma \sin\theta - \omega_d \cos\theta)$

let us set $\rho = A e^{-\gamma t}$

$$\Rightarrow \begin{cases} x = \rho \sin\theta \\ \dot{x} = -\rho(\gamma \sin\theta - \omega_d \cos\theta) \end{cases} \quad (4)$$

We can use the linear transformation

$$y = \dot{x} + \gamma x$$

which means that

$$y = -\rho(\gamma \sin\theta - \omega_d \cos\theta) + \gamma\rho \sin\theta$$

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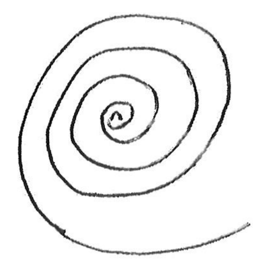
$$\Rightarrow \underline{\underline{y = \omega_d \rho \cos\theta}}$$

Therefore (4) becomes $\begin{cases} x = \rho \sin\theta \\ y = \omega_d \rho \cos\theta \end{cases}$ (5)

$$y^2 = \omega_d^2 \rho^2 \cos^2\theta = \omega_d^2 \rho^2 (1 - \sin^2\theta)$$

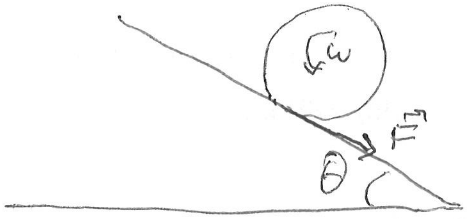
$$\Rightarrow \frac{y^2}{\omega_d^2 \rho^2} = \cos^2\theta . \qquad \frac{x^2}{\rho^2} = \sin^2\theta$$



$$\Rightarrow \boxed{\frac{x^2}{\rho^2} + \frac{y^2}{\omega_d^2 \rho^2} = 1}$$

The trajectory of the underdamped oscillator in modified phase space (x, y) is an ellipse whose major and minor axes decrease exponentially (see $\rho = A e^{-\gamma t}$)

CM (B4)



before pure rolling the friction force is

$$F = \mu m g \cos\theta$$

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equations of motion

$$m\dot{v} = -mg\sin\theta - \mu m g\cos\theta \quad (1)$$

$$I\dot{\omega} = F_p r \quad (2)$$ where r is the radius of the ball

(a) Solve (1): $v = v_0 - g(\sin\theta + \mu\cos\theta)t$

$$S = v_0 t - g(\sin\theta + \mu\cos\theta)\frac{t^2}{2} \quad (3)$$

(b) From Eq. (2)

$$\omega = \frac{\mu m g\cos\theta}{I} r t = \frac{5}{2}\frac{\mu g\cos\theta}{r} t$$

where we used $I = \frac{2}{5} m r^2$

slipping stops when $v = r\omega$ — 5

$$v_0 - g(\sin\theta + \mu\cos\theta)t = \frac{5}{2}\mu g\cos\theta\, t$$

Solve for t

$$t_0 = \frac{v_0}{g(\sin\theta + \frac{7}{2}\mu\cos\theta)}$$

(c) substitute t_0 in (3)

$$S = \frac{v_0^2}{g(\sin\theta + \frac{7}{2}\mu\cos\theta)} - g(\sin\theta + \mu\cos\theta)\frac{v_0^2}{2g^2(\sin\theta + \frac{7}{2}\mu\cos\theta)^2}$$

$$= \frac{v_0^2}{g(\sin\theta + \frac{7}{2}\mu\cos\theta)^2}\left(\sin\theta + \frac{7}{2}\mu\cos\theta - \frac{1}{2}\sin\theta - \frac{1}{2}\mu\cos\theta\right)$$

$$= \frac{v_0^2(\frac{1}{2}\sin\theta + 3\mu\cos\theta)}{g(\sin\theta + \frac{7}{2}\mu\cos\theta)^2}$$ 5

(d) $t < t_0$: no since mech. energy is dissipated | $t > t_0$ yes friction works for rot. energy 5

Preliminary Thermal – August 2024

Easy Problems: A1

1. One mole of van de Waals gas $(P + a/V^2)(V - b) = RT$ expands from V_i to V_f isothermally, how much is the work done by the system.

   Solution:

$$P = RT/(V - b) - a/V^2$$

$$W = \int PdV = \int \left[\frac{RT}{V - b} - \frac{a}{V^2}\right] dV$$

$$= RT\, ln \left[\frac{V_f - b}{V_i - b}\right] + a(1/V_f - 1/V_i)$$

2. A2 Show that, for ideal gas C_P-C_V=$\alpha_V^2\, TV/K_T$ where the thermal expansion coefficient $\alpha = (\partial V/\partial T)_P/V$ and isothermal compressibility $K_T = -(\partial V/\partial P)_T/V$

   Solution:

   For ideal gas,

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{nR}{PV} = \frac{1}{T}$$

$$K = -1/V\left(\frac{\partial V}{\partial P}\right)_T = 1/P$$

$$\text{So } \alpha_V^2 TV/K = \left(\frac{1}{T}\right)^2 PTV = \frac{PV}{T} = nR$$

3. A3 A polytropic process is a thermodynamic process that obeys the relation: $PV^l = C$, where C is a constant. Show that heat capacity in a polytropic process is $C_l = \lim(Q/\Delta T)_{\text{polytropic}} = C_V + nR/(1-l)$.

   Solution:

   Given $PV^l = C$ and $PV = nRT$, one can derive $V = \left(\frac{C}{nRT}\right)^{\frac{1}{l-1}}$, $P = \frac{(nRT)^{\frac{l}{l-1}}}{C^{\frac{1}{l-1}}}$.

$$\Delta V = \left(\frac{C}{nRT}\right)^{\frac{1}{l-1}} \left(\frac{-1}{l-1}\right)\frac{\Delta T}{T},\ P\Delta V = \frac{nR}{1-l}\,\Delta T$$

   The heat capacity $C_l = \lim_{\Delta T \to 0}\left(\frac{Q}{\Delta T}\right)_{polytropic} = \lim_{\Delta T\to 0}\frac{\Delta U + P\Delta V}{\Delta T}$

$$= \lim_{\Delta T\to 0}\frac{\Delta U}{\Delta T} + \lim_{\Delta T\to 0}\frac{P\Delta V}{\Delta T}$$

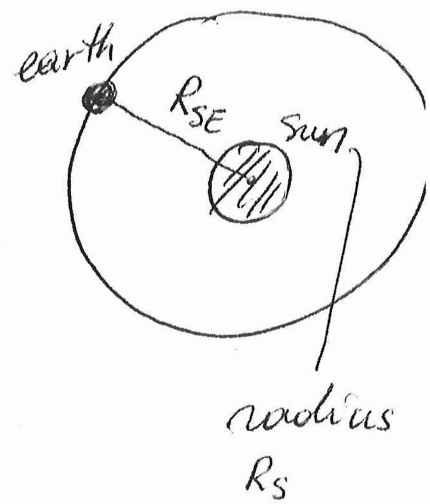
$$= C_V + \frac{nR}{1-l}$$

(1)

Thermo (A4)
Preliminary Exam Key

August 2024
Marcel

pb #1: (thermo - Easy)



$$0.1\ W/m^2 = \frac{P_{\text{emitted by sun}}}{4\pi R_{SE}^2}$$

where

$$P_{\text{emitted by sun}} = J_S A_S$$

with $J_S = \sigma T^4$ (radiant flux density of the sun, black body)

$$\sigma = 5.7 \times 10^{-8}\ W/m^2 \cdot K^{-4}$$

and A_S = surface of the sun $= 4\pi R_S^2$

$$\Rightarrow 0.1\ W/m^2 = \sigma T^4 \left(\frac{R_S}{R_{SE}}\right)^2$$

$$\Rightarrow T = \left[\frac{0.1\ \text{x10^4 W/m^2}}{\sigma}\left(\frac{R_{SE}}{R_S}\right)^2\right]^{1/4} \simeq 5.3 \times 10^3\ K$$

$R_S = 7 \times 10^8\ m$, $R_{SE} = 1.5 \times 10^{11}\ m$

Hard Problems:

1. B1 One mole of diatomic ideal gas (C_V = 2.5 nR) performs a transformation from an initial state with temperature of 291 K and volume of 21,000 ml to a final state with temperature of 305 K and volume of 12,700 ml. The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

   Solution:

   Using $PV = N_A k_B T$:

	Initial	**Final**	**Change**
P (Pa)	$1.15\text{x}10^5$	$2\text{x}10^5$	0.85E5
V (m^3)	**$2.1\text{x}10^{-2}$**	**$1.27\text{x}10^{-2}$**	$-8.3\text{x}10^{-3}$
T (K)	**291**	**305**	14

   A straight line in (P,V) diagram: $P = P_1 + \frac{\Delta P}{\Delta V}(V - V_1)$,

   $W = \int PdV = \int \left[P_1 + \frac{\Delta P}{\Delta V}(V - V_1)\right] dV = \left(P_1 - \frac{\Delta P}{\Delta V}V_1\right)\Delta \mathrm{V} + \frac{1}{2}\Delta P(V_1 + V_2)$ =-1307 J

   $\Delta U = \mathrm{C_V}\Delta T = 290\,J$

   $Q = \Delta U + W = -1017\,J$

2. B2 Consider a wire with equation of state: $F = bT\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right)$, where L is the length, $L_0(T)$ is the length when the tension F is zero, b is a constant. Like the PVT system, one can define linear thermal expansion coefficient $\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_F$, isothermal Young's modulus $Y = \frac{L}{A}\left(\frac{\partial F}{\partial L}\right)_T$, where A is the area of the cross section of the wire.

   a. Show that $Y = \frac{bT}{A}(\frac{L}{L_0} + \frac{2L_0^2}{L^2})$.

   b. Find Y_0 for F = 0.

   c. For a rubber band at T=300 K, b =1.33×10^{-2} N/K, A =1×10^{-6} m^2, calculate the value of F and Y for $\frac{L}{L_0}$ = 0.5, 1.0, 1.5 and 2.

   Solution:

   a. $F = bT\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right)$

   $\left(\frac{\partial F}{\partial L}\right)_T = bT\left(\frac{L}{L_0} + \frac{2L_0^2}{L^2}\right)$

   $Y = \frac{L}{A}\left(\frac{\partial F}{\partial L}\right)_T = \frac{bT}{A}\left(\frac{L}{L_0} + \frac{2L_0^2}{L^2}\right)$

   b. When F = 0, $L = L_0$. $Y = \frac{3bT}{A}$.

   c.

L/L_0	**0.5**	**1.0**	**1.5**	**2.0**
F (N)	-14	0	4.2	7.0
Y(N/m^2)	3.4x10^7	1.2x10^7	9.5x10^6	1x10^7

Thermo B3

One mole of a monoatomic gas is expanded adiabatically amd quasi-statically from an initial pressure of 2 atm and temperature 300 K to a final pressure 1 atm. Find

(a) The final volume

(b) The final temperature

(c) the work done by the gas.

(d) The change in the internal energy of the gas.

Solution (a)

$$V_1 = \frac{RT_1}{p_1} = \frac{0.08206 \cdot 300}{2} = 12.31L \qquad \textbf{5points}$$

where we use $R = 0.08206$ L·atm/(mol·K) derived from 1 atm·L=101325 Pa·0.001 m^3=101.325 J and R=8.3145 J/(mol·K). (Alternatively they can use SI units by converting atm into Pa. The answer is $V_1 = 0.01231$ m^3.)

For adiabatic process

$$P_1V_1^\gamma = P_2V_2^\gamma$$

where $\gamma = (c_V + R)/c_V = 5/3$ for a monoatomic gas.

$$V_2 = V_1\left(\frac{P_1}{P_2}\right)^{1/\gamma} = V_1 2^{3/5} = 18.66 \text{ L} \qquad \textbf{7points}$$

(b)

$$T_2 = \frac{P_2V_2}{R} = \frac{1 \cdot 18.66}{0.08206} = 227 \text{ K.} \qquad \textbf{5points}$$

(c)

$$W = -c_V\Delta T = -\frac{3}{2}R \cdot (-73) = 910 \text{ J} \qquad \textbf{5points}$$

(d)

$$\Delta U = -W = -910 \text{ J.} \qquad \textbf{3points}$$

Thermo

(B4) solution

$$dS = \frac{dQ(x)}{T(x)}$$

initial temperature $T_i(x) = T_1 + (T_2 - T_1)x/L$

For dx section of the rod

$$dS = C_p \frac{dx}{L} \int_{T_i(x)}^{(T_1+T_2)/2} \frac{dT}{T} = C_p \frac{dx}{L} \ln \frac{(T_1+T_2)/2}{T_1 + (T_2-T_1)x/L}$$

The total change of S

$$\Delta S = \frac{C_p}{L} \int_0^L dx \ln \frac{(T_1+T_2)/2}{T_1 + (T_2-T_1)x/L}$$

The first integral is

$$C_p \ln[(T_1+T_2)/2]$$

For the second integrate by parts

$$-\int_0^L dx \ln\left[T_1 + (T_2-T_1)\frac{x}{L}\right] = -x \ln\left(T_1 + (T_2-T_1)\frac{x}{L}\right)\Big|_0^L$$

$$+ \int_0^L \frac{x\,dx}{T_1 + (T_2-T_1)\frac{x}{L}} \frac{T_2-T_1}{L} = -L \ln T_2 + \int_0^L \frac{x\,dx}{\frac{LT_1}{T_2-T_1} + x} =$$

$$= -L \ln T_2 + L - \frac{LT_1}{T_2-T_1} \int_0^L \frac{dx}{x + \frac{LT_1}{T_2-T_1}} = -L \ln T_2 + L - \frac{LT_1}{T_2-T_1} \ln \frac{L + \frac{LT_1}{T_2-T_1}}{\frac{LT_1}{T_2-T_1}}$$

$$= -L \ln T_2 + L - \frac{LT_1}{T_2-T_1} \ln \frac{T_2}{T_1}$$

summing two integrals, we obtain

$$\Delta S = C_p \left[\ln \frac{T_1+T_2}{2} - \ln T_2 + 1 - \frac{T_1}{T_2-T_1} \ln \frac{T_2}{T_1}\right]$$

$$= C_p \left[1 + \ln \frac{T_1+T_2}{2T_2} - \frac{T_1}{T_2-T_1} \ln \frac{T_2}{T_1}\right]$$

Grading Rubric for Thermal Physics (Preliminary Exam, August 2024)

NB: If there is a “no errors” entry for a problem, minor errors should only count against that budget.

A1

$W = \int p dV$	5 points
Correctly express $p(V)$ and insert it in the integral	5 points
Carry out the indefinite integral, perhaps with minor errors	6 points
Substitution of limits	4 points
No errors in the integration	5 points

A2

Equation of state (EOS) for ideal gas	2 points
Calculate α from the EOS	6 points
Calculate K from the EOS	6 points
Substitute and verify the identity	5 points
No errors	6 points

A3

State the definition of heat capacity $C = \delta Q/\delta T$	2 points
State first law $\delta Q = dU + pdV$	2 points
Ideal gas $pV = nRT$	1 points
Ideal gas $C_V = dU/dT$	2 points
Combine EOS with $pV^l = C$ to get p as a function of T only	4 points
Combine EOS with $pV^l = C$ to get V as a function of T only	4 points
Calculate $\frac{pdV}{dT} = nR/(1-l)$	4 points
Put everything together	1 points
No errors	5 points

A4

Flux from unit surface area of the Sun through σ	6 points
Total flux from the Sun including $4\pi R_S^2$	8 points
Flux per unit area at earth (divide by $4\pi R_{orb}^2$)	8 points
Substitute numbers and compute the answer	1 point
Reasonable order of magnitude	2 points

B1

Equation $p(V)$ for the straight line	5 points
Set up integral for the work	3 points
Carry out the integration	5 points
Substitute numbers and calculate the work	1 point
First law (connect heat with energy and work)	2 points
Relation of $\Delta U = C_V \Delta T$	3 points
Calculate ΔU	1 point
Add the two to get heat absorbed	1 point
No errors in the formulas	4 points
Correct numerical results	0 points

B2

Derive *Y*	5 points
Show that F=0 at $L=L_0$	4 points
Substitute $L=L_0$ into *Y* to find Y_0	4 points
Calculate the values of F and Y for different L/L0 (general approach)	5 points
Correct numerical values (1 point for each L/L0)	4 points
No errors in the formulas	3 points

B4

$dS = \delta Q/T$	3 points
Argue that the final temperature T_f is the average of $T(x)$	2 points
Set up integral for entropy change for a piece of length dx	3 points
Integrate it from $T(x)$ to T_f	4 points
Set up integral over the length of the wire	4 points
Carry out the integral over the length (integration by parts)	4 points
No errors	5 points