UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 16, 2024

This test covers the topic of *Classical Mechanics.* There are 4 “A” questions and 4 “B” questions.Work **two** problems from each group. Thus, you will work on a total of 4 questions on this topic, 2 from each group.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE SCRATCH PAPER ONLY

Classical Mechanics Group A

Answer only two Group A questions

A1. Consider a two-dimensional oblique elastic collision between two identical billiard balls on a frictionless table, where one ball is initially stationary. Prove that the two balls emerge from collision at the right angle to each other.

A2. A gun is fired horizontally at a block of wood of mass 1 kg which is initially at rest on a horizontal floor. The bullet of mass 10 g becomes imbedded in the block, and the impact causes the system to slide the distance 4 m before coming to rest. The initial speed of the bullet is 430 m/s. Find the coefficient of sliding friction between the block and the floor.

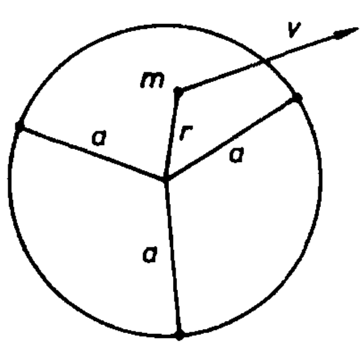
A3. The distance of the Halley's comet from the sun at the perihelion is 8.800x10^7 km, and its speed at the perihelion is 54.47 km/s. Find its speed when it is at the distance 2.000x10^9 km from the sun. Use GM=1.327x10^{20} m^3/s^2 where M is the sun's mass. Obtain the answer with two-digit accuracy. Explain why four-digit accuracy in calculation is necessary for this.

A4. Show that the kinetic energy of a two-particle system is $mv_{cm}^2/2+\mu v^2/2$ where $m=m_1+m_2$, v is the relative speed, v_{cm} is the speed of the center of mass, and μ is the reduced mass.

Classical Mechanics Group B

Answer only two Group B questions

B1. Three fixed point particles are equally spaced about the circumference of a circle of diameter $2a$ centered at the origin, see figure below. The force exerted by each particle on a point mass m is attractive and given by $\boldsymbol{F} = -k\boldsymbol{R}$, where $\boldsymbol{R}$ is a vector drawn from the particle to the point mass. The point mass is placed in the force field at time $t = 0$ with initial conditions $\boldsymbol{r} = \boldsymbol{r}_0$ and $\dot{\boldsymbol{r}} = \boldsymbol{v}_0$.



a) Define suitable coordinates and write down an expression for the force acting on the mass at any time.

b) Use Newton's second law and solve the equation of motion for the initial conditions given above, namely, find $\boldsymbol{r}(t)$ in terms of $\boldsymbol{r}_0$, $\boldsymbol{v}_0$ and the parameters of the system.
c) Under what conditions, if any, are circular orbits a solution?

B2. A bead slides on a uniformly rotating straight horizontal wire in a force-free space. The angular frequency of the rotation of the wire about a vertical axis is ω. The bead is initially at rest on the wire at a distance r_0 from the rotation axis. Neglect the mass of the wire.

a) Find a suitable generalized coordinate for the system and write down the Lagrangian of the system.
b) Find the equation of motion.
c) Solve the equation of motion and show that the bead's position grows like $\cosh\omega t$ with time t.
d) Use the 2^{nd} Newton law for rotational motion to determine the force of constraint that keeps the bead on the wire.

B3. Consider the motion of an underdamped 1D harmonic oscillator subject to a drag force $-c\dot{x}$, where $c > 0$ is a constant.

a) Write down the equation of motion and find the solution of the problem in terms of $\gamma = c/2m$, $\omega_0 = \sqrt{k/m}$, as well as the amplitude and phase constants A and ϕ.
b) Define a modified phase space by introducing coordinates (x,y) where $y=dx/dt+\gamma x$. Show that the trajectory of the underdamped oscillator in this phase space is an ellipse whose major and minor axes decrease exponentially with time.

B4. A ball is projected, initially without rotation, at a speed v_0 up a rough inclined plane of inclination θ and coefficient of sliding friction μ.

(a) Find the position of the ball as a function of time before pure rolling begins.
(b) Determine the instant t_0 when pure rolling begins.
(c) Find the position of the ball when pure rolling begins.
(d) Is mechanical energy conserved at $t < t_0$? At $t > t_0$?

Physical Constants

Speed of light .... $c = 2.998\times10^{8}$ m/s
Atmospheric pressure........... $101{,}325$ Pa
Electron mass .... $m_e = 9.109\times10^{-31}$ kg
Avogadro constant .... $N_\text{A} = 6.022\times10^{23}$ mol^{-1}
Boltzmann constant.............. $k_\text{B} = 1.381\times10^{-23}$ J / K = 8.617×10^{-5} eV/K
Gas constant .... $R = 8.314$ J / (mol·K)
Atomic mass unit .... $1\text{u} = 1.66\times10^{-27}$ kg
Gravitational constant .... $G = 6.674\times10^{-11}$ m^3 / (kg·s^2) ; $g = 9.8$ m/s^2

Equations That May Be Helpful

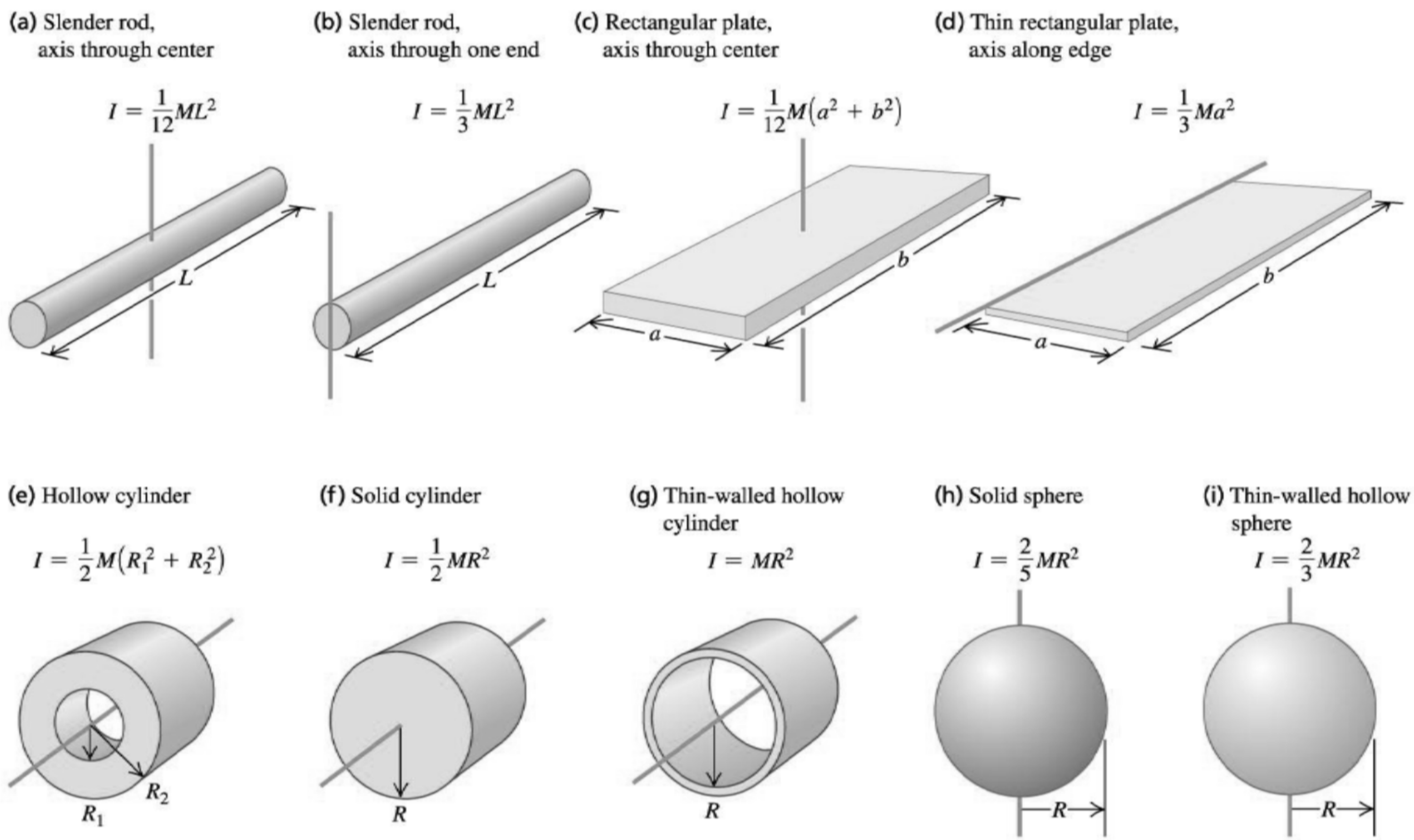
TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

TABLE 9.2 Moments of Inertia of Various Bodies



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$\int_0^\infty \frac{1}{1+bx^2}\,dx = \frac{\pi}{2b^{1/2}}$	$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$
$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$	$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2a}$
$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$	$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$
$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$	$\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2a^2}$
$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$	$\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$
$\int (x^2+b^2)^{-2}\,dx = \frac{\frac{bx}{x^2+b^2}+\arctan\left(\frac{x}{b}\right)}{2b^3}$	$\int_0^\infty x^5 e^{-x^2}\,dx = \frac{1}{a^3}$
$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$	$\int_0^\infty x^6 e^{-x^2}\,dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$
$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$	
$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) =$	
$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$	

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WRITE YOUR ANSWERS ON ONE SIDE OF THE SCRATCH PAPER ONLY

Thermodynamics and Statistical Mechanics Group A
Answer only two Group A questions

A1. One mole of van der Waals gas, whose equation of state is $(P + a/V^2)(V - b) = RT$, expands from volume V_i to V_f isothermally. Calculate the work done by the gas.

A2. Show that for an ideal gas C_P-C_V=$\alpha^2 TV/K_T$ where the thermal expansion coefficient $\alpha = (\partial V/\partial T)_P/V$ and isothermal compressibility $K_T = -(\partial V/\partial P)_T/V$.

A3. A polytropic process is a thermodynamic process that obeys the relation: $PV^l = C$, where C is a constant. Show that heat capacity of an ideal gas in a polytropic process is

$C_l = \lim(Q/\Delta T)_{\text{polytropic}} = C_V + nR/(1-l)$.

A4. The solar constant (radiant flux at the surface of the earth) is about 0.1 W/cm^2. Find the temperature of the sun's surface assuming that it is a black body. The radius of the sun is 7×10^5 km, and the distance from the sun to the earth is 1.5×10^8 km.

Thermodynamics and Statistical Mechanics Group B
Answer only two Group B questions

B1. One mole of diatomic ideal gas (C_V = 2.5 nR) performs a transformation from an initial state with temperature of 291 K and volume of 21,000 ml to a final state with temperature of 305 K and volume of 12,700 ml. The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

B2. Consider a wire with equation of state: $F = bT\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right)$, where L is the length, $L_0(T)$ is the length when the tension F is zero, b is a constant. Like for the PVT system, one can define linear thermal expansion coefficient $\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_F$, isothermal Young's modulus $Y = \frac{L}{A}\left(\frac{\partial F}{\partial L}\right)_T$, where A is the area of the cross section of the wire.

a. Show that $Y = \frac{bT}{A}\left(\frac{L}{L_0} + \frac{2L_0^2}{L^2}\right)$.
b. Find Y_0 for F = 0.
c. For a rubber band at T=300 K, b =1.33×10^{-2} N/K, A =1×10^{-6} m^2, calculate the value of F and Y for $\frac{L}{L_0}$ = 0.5, 1.0, 1.5 and 2.

B3. One mole of a monoatomic gas is expanded adiabatically amd quasi-statically from an initial pressure of 2 atm and temperature 300 K to a final pressure 1 atm. Find

(a) The final volume.

(b) The final temperature.

(c) The work done by the gas.

(d) The change in the internal energy of the gas.

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B4. Consider a homogeneous rod with a linear temperature distribution

$$T(x) = T_1 + (T_2 - T_1)x/L$$

in which the temperatures at the two ends are T_1 and T_2. Show that after the rod reaches thermal equilibrium (with temperature $\frac{T_1+T_2}{2}$), the change of the total entropy is

$$\Delta S = C_p \left(1 + \ln \frac{T_1 + T_2}{2T_2} - \frac{T_1}{T_2 - T_1} \ln \frac{T_2}{T_1}\right)$$

where C_p is the heat capacity.

Physical Constants

Speed of light .... $c = 2.998 \times 10^{8}$ m/s
Atmospheric pressure........... 101,325 Pa
Electron mass .... $m_e = 9.109 \times 10^{-31}$ kg
Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}
Boltzmann constant.............. $k_B = 1.381 \times 10^{-23}$ J / K = 8.617×10^{-5} eV/K
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Atomic mass unit .... 1u $= 1.66 \times 10^{-27}$ kg
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Equations That May Be Helpful

TRIGONOMETRY

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$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
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For small x:

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$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

THERMODYNAMICS

Heat capacity $C_V = N\frac{d\langle E\rangle}{dT}$.

Molar heat capacity of diatomic gas: $C_V = \frac{5}{2}R$.

For adiabatic processes in an ideal gas with constant heat capacity, $pV^\gamma = \text{const}$.

$$dU = TdS - pdV$$
$$dF = -SdT - pdV$$

$$H = U + pV$$
$$F = U - TS$$
$$G = F + pV$$
$$\Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$
$$C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$
$$TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$
$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$

Carnot engine: ΔS=0

Carnot efficiency = 1 – T_c/T_h .

The cyclic rule: $\left(\frac{\partial T}{\partial P}\right)_H \left(\frac{\partial P}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_P = -1.$

Stefan-Boltzmann's law:

$$P = \sigma T^4; \quad \sigma = 5.67\times 10^{-8} W\cdot m^{-2}\cdot K^{-4}$$

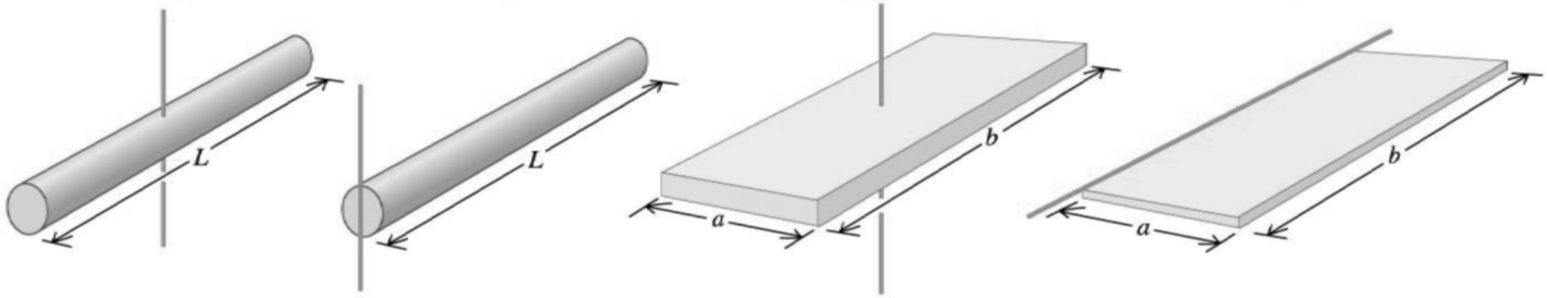
TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center: $I = \frac{1}{12}ML^2$

(b) Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

(c) Rectangular plate, axis through center: $I = \frac{1}{12}M(a^2 + b^2)$

(d) Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



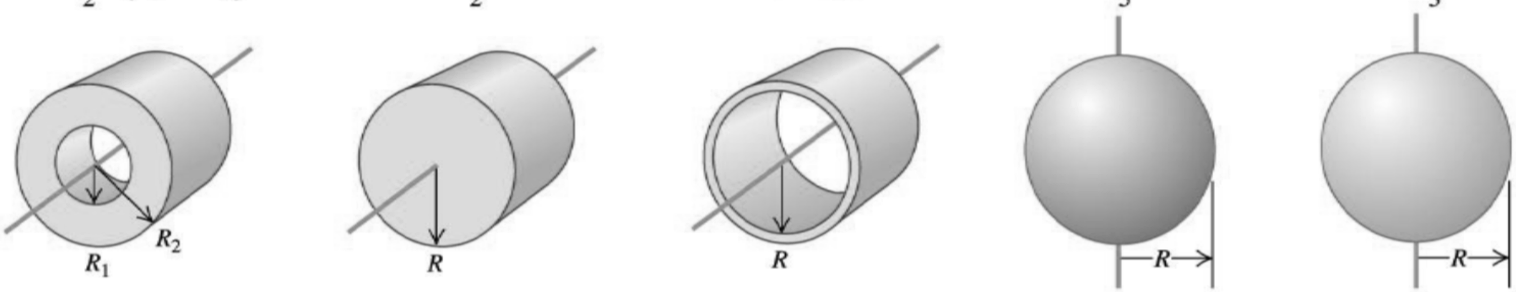
(e) Hollow cylinder: $I = \frac{1}{2}M(R_1^2 + R_2^2)$

(f) Solid cylinder: $I = \frac{1}{2}MR^2$

(g) Thin-walled hollow cylinder: $I = MR^2$

(h) Solid sphere: $I = \frac{2}{5}MR^2$

(i) Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$\int_0^\infty \frac{1}{1+bx^2}dx = \frac{\pi}{2b^{1/2}}$	$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$
$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$	$\int_0^\infty x e^{-x^2}dx = \frac{1}{2a}$
$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$	$\int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{2a^{3/2}}$
$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$	$\int_0^\infty x^3 e^{-x^2}dx = \frac{1}{2a^2}$
$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$	$\int_0^\infty x^4 e^{-x^2}dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$
$\int (x^2+b^2)^{-2}dx = \frac{\frac{bx}{x^2+b^2}+\arctan\left(\frac{x}{b}\right)}{2b^3}$	$\int_0^\infty x^5 e^{-x^2}dx = \frac{1}{a^3}$
$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$	$\int_0^\infty x^6 e^{-x^2}dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$
$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$	
$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\text{artanh}\left(\frac{ax}{b}\right)$	