UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Friday, August 9, 2024

This test covers the topics of *Electrodynamics*. There are 4 “A” questions and 4 “B” questions. Work **two** problems from each group. Thus, you will work on a total of 4 questions on this topic, 2 from each group.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

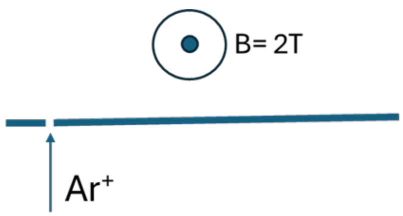
Electrodynamics Group A

Answer only two Group A questions

A1. A spin 1/2 neutral Cu atom beam (copper m=1.05x10^{-25} kg) enters a Stern-Gerlach experiment 1 m long with a filed gradient $\partial B_z/\partial z$ of 12 T/m. The moment on each spin 1/2 atom is 9.274x10^{-24} J/T and the atoms enter with velocity 500 m/s.

a) If the direction motion along the Stern-Gerlach is 'x', what is the velocity of the atoms along the field gradient z, at the end of the Stern-Gerlach apparatus ?
b) How big is the deflection at the end of the Stern-Gerlach experiment?

A2. Singly positive ions of Ar with the mass number 40 enter a vacuum chamber vertically through a slit in a horizontal plate, after having been accelerated through a voltage of 20.0 kV. The path is then bent by a magnetic field whose direction is parallel to the plate and magnitude is 2 T. Find the distance from the slit where ions are deposited on the plate.



A3. A metal disk whose conductivity is σ is located in the *xy* plane. There is a time-dependent uniform magnetic field $B(t)=B_0\exp(-t/\tau)$ directed along the z axis. Determine the induced current density in the disk (magnitude and direction) as a function of time t and the distance r from the disk center.

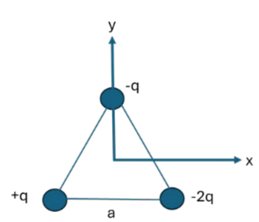
A4. The amplitude of the electric field oscillation in a radio wave is 0.005 V/m.

(a) What is the amplitude of magnetic field oscillations?
(b) What is the wave's intensity?

Electrodynamics Group B

Answer only two Group B questions

B1. Three point charges $+q$, $-q$, and $-2q$ are placed at the corners of an equilateral triangle with the side length a.

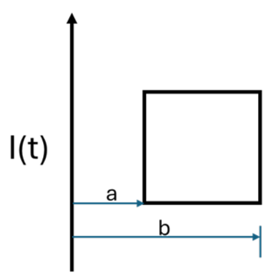


(a) What is the electric potential and the electric field (magnitude and direction) at the center of the triangle?
(b) Find the work required to increase the side of the triangle from a to $2a$. Is it positive or negative?

B2. A point electric dipole is located at the point $(0,0,z_0)$ above a grounded (conducting) plane z=0. The dipole moment vector **p** is parallel to the plane.

(a) Find the electric field $\mathbf{E}(0,0,z)$ on the z axis.
(b) Find the asymptotic expression for **E** when $z \gg z_0$. What kind of multipole is it?.

B3. A long straight wire carries an alternating current $I(t)=I_0\cos(\omega t)$. Nearby is a square loop. The wire lies in the plane of the loop, parallel to two sides of the square, which are of distances a and b from the wire. (The side of the square is b-a).



(a) Find the current in the square loop if its resistance is R
(b) Find the direction of the current in the loop at $t=\pi/(2\omega)$ and $t=3\pi/(2\omega)$

B4. (i) Find the bound charge density of a uniformly polarized sphere whose polarization is given by a constant vector **P.** (It is convenient to direct this vector along the z axis).

(ii) Using the result obtained in (i) calculate the potential for the polarized sphere of radius a.

Physical constants

Speed of light .... $c = 2.998\times10^{8}$ m/s
Planck's constant .... $h = 6.626\times10^{-34}$ J·s
Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s
Electron mass .... $m_e = 9.109\times10^{-31}$ kg
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Atomic mass unit .... 1 u=1.66 × 10^{-27} kg
Electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m
Bohr radius.................... $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e}$= 0.5292 Å
Magnetic permeability.... $\mu_0 = 1.257\times10^{-6}$ H/m
Rydberg unit of energy ... $Ry = 13.6$ eV
Rydberg constant……..... R=1.097x10^{7} m^{-1}
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Molar gas constant.......... $R = 8.314$ J / mol·K
Gravitational constant...... $G = 6.674\times10^{-11}$ $\mathrm{m}^3/\mathrm{kg\,s}^2$
Avogadro constant .... $N_A = 6.022\times10^{23}$ mol^{-1}
hc .... $hc = 1240$ eV·nm
Fine structure constant .. $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$
$E^2=p^2c^2+m^2c^4$

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$
$$\sin\alpha\sin\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
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$$\cos(ix)=\cosh(x)$$
$$\sin(ix)=i\sinh(x)$$

For small x:

$$\sin x\approx x-\tfrac{1}{6}x^3$$
$$\cos x\approx 1-\tfrac{1}{2}x^2$$
$$\tan x\approx x+\tfrac{1}{3}x^3$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da=\frac{q_{\text{encl}}}{\varepsilon_0}\,;\qquad \mathbf{E}=-\nabla\Phi\,;\qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{E}\cdot d\boldsymbol{\ell}=\Phi(\mathbf{r}_1)-\Phi(\mathbf{r}_2)\,;\qquad \Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,.$$

Work done: $W=-\int_{\mathbf{a}}^{\mathbf{b}}q\mathbf{E}\cdot d\boldsymbol{\ell}=q\left[\Phi(\mathbf{b})-\Phi(\mathbf{a})\right]$.

Energy stored in electric field: $W=\tfrac{1}{2}\varepsilon_0\int_V E^2\,d\tau=Q^2/2C$.

Multipole expansion: $\Phi(\mathbf{r})=\dfrac{q}{4\pi\varepsilon_0 r}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{\mathbf{r}\cdot\mathbf{p}}{r^3}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{2}\sum_{ij}Q_{ij}\dfrac{x_i x_j}{r^5}+\ldots$

Field of electric dipole:

$$\mathbf{E}(\mathbf{r})=\frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{4\pi\epsilon_0 r^3}$$

Monopole moment: $q=\int\rho(\mathbf{r})d^3\mathbf{r}$.

Dipole moment: $\mathbf{p}=\int\rho(\mathbf{r})\mathbf{r}\,d^3\mathbf{r}$.

Quadrupole moment : $Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right]d^3\mathbf{r}$ (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$) .

Parallel-plate capacitor: $C = \varepsilon_0 \dfrac{A}{d}$.

Spherical capacitor: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$.

Cylindrical capacitor: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L).

Relative permittivity: $\varepsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}}$.

Bound charges: $\rho_{\mathrm{b}} = -\nabla\cdot\mathbf{P}$; $\sigma_{\mathrm{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$.

MAGNETOSTATICS

Relative permeability: $\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}}$.

Lorentz force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$.

Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$.

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\int \dfrac{I d\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$) .

B-field inside of an infinitely long solenoid: $\mathbf{B} = \mu_0 n I \hat{\boldsymbol{\varphi}}$ (n is the number of turns per unit length).

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\mathrm{encl}}$.

Magnetic dipole moment of a planar current distribution: $\mathbf{m} = I\int d\mathbf{a}$.

Force on a magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$.

Torque on a magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$.

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}$.

Bound currents: $J_{\mathrm{b}} = \nabla\times\mathbf{M}$; $K_{\mathrm{b}} = \mathbf{M}\times\hat{\mathbf{n}}$.

Maxwell's equations in vacuum

$\nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss' law
$\nabla\cdot\mathbf{B} = 0$	no magnetic charge
$\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$	Faraday's law
$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \varepsilon_0\mu_0\dfrac{\partial\mathbf{E}}{\partial t}$	Ampere's law with Maxwell's correction

Maxwell's equations in linear, isotropic, and homogeneous media

$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$	Gauss' law
$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \dfrac{\partial \mathbf{D}}{\partial t}$	Ampere's law with Maxwell's correction

Alternative way of writing Faraday's law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\mathsf{F}_B}{dt}$.

Mutual and self inductance: $\mathsf{F}_2 = M_{21} I_1$; $\mathsf{F} = LI$.

Energy stored in magnetic field: $W = \tfrac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$.

Wave equations in a conducting medium:

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}; \quad \nabla^2 \mathbf{B} = \mu\sigma \frac{\partial \mathbf{B}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda; \quad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}.$$

Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0.$$

Lorenz gauge:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0.$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int_0^\infty \frac{1}{1+bx^2}\,dx = \frac{\pi}{2b^{1/2}}$$

$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$$

$$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}\,dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan\left(\dfrac{x}{b}\right)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) =$$

$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$

$$\int x^4 e^{-x}\,dx = -e^{-x}\left(x^4+4x^3+12x^2+24x+24\right)$$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

$$\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_0^\infty x^5 e^{-x^2}\,dx = \frac{1}{a^3}$$

$$\int_0^\infty x^6 e^{-x^2}\,dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$$

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<u>Quantum Mechanics Group A</u>

Answer only two Group A questions

A1. A photon and an electron have the same wavelength. The electron has a kinetic energy of K. What is the photon's energy in terms of K

(a) For the non relativistic case
(b) for the relativistic case
(c) Obtain a numerical result for the photon energy when K=1 MeV.

A2. Consider an experiment in which a beam of electrons is directed at a plate containing two slits, labelled A and B, separated by a distance d. Beyond the plate is a screen equipped with an array of detectors which enables one to determine where the electrons hit the screen.
a) Draw a rough graph of the relative number of incident electrons as a function of position along the screen for the case that Slit A open, slit B closed.

b) Let's call the probability of detecting electrons passing through slit A at the screen $I_A(x)$ (with slit B closed) and the probability of detecting electrons passing through slit B at the screen $I_B(x)$ (with slit A closed) , and the probability of detecting electrons at the screen when both slits are open $I_{AB}(x)$. Does $I_{AB}(x) = I_A(x) + I_B(x)$? Why or why not?

c) Now consider the case that Stern-Gerlach apparatus attached to the slits in such a manner that only electrons with

$$s_z = \hbar/2$$

can pass through A and only electrons with

$$s_z = -\hbar/2$$

can pass through B. What is the probability of detecting electrons at the screen in terms of $I_A(x)$ and $I_B(x)$? Explain your reasoning.

A3. Consider a one dimensional linear harmonic oscillator and let ψ_0 and ψ_1 be its real, normalized ground and first excited state eigenfunctions respectively. Let $A\psi_0+B\psi_1$ with A and B real numbers, be the wave function of the oscillator at some instant of time.
(a)Show that the average value of x, denoted as $\langle x \rangle$ is in general different from zero.
(b) What values of A and B maximize $\langle x \rangle$ and what values minimize it?

A4. A muon is a negatively charged particle whose charge is the same as that of electron, and the mass is $207m_e$. Consider a muonium atom consisting of a proton and a muon.

(a) Describe the difference between the muonium aton and the hydrogen atom.
(b) Find the energy of the ground state and the first excited state.
(c) What is the wavelength of the photon emitted when the atom makes a transition from the first excited state to the ground state?

Quantum Mechanics Group B
Answer only two Group B questions

B1. An electron is confined in a three-dimensional infinite potential well. The sides parallel to the x, y, and z axes are of length L each.
(a) Write down the *time dependent* Schroedinger equation.
(b) Write down the wave function corresponding to the state of the lowest possible energy.
(c)Which is the lowest energy state with a degeneracy >3? Write out all possible combinations of the quantum numbers n_x, n_y and n_z and the energy E_n.

B2. Consider a system of spin l/2.
(a)What are the eigenvalues and normalized eigenvectors of the operator $As_y + Bs_z$, where s_y and s_z are the angular momentum operators, and A and B are real constants.

(b) Assume that the system is in a state corresponding to one of the eigenstates (either one is fine, just specify one). What is the probability that a measurement of s_y, will yield the value $+\hbar/2$?

B3. A particle of charge q and mass m in one dimension is subject to a uniform electrostatic field E. The Hamiltonian is given by $\frac{p^2}{2m} - qE \cdot r$.

(a)Write down the time-dependent Schroedinger equation for this system.

(b) Show that the expectation value of the position operator obeys Newton's second law of motion when the particle is in an arbitrary state $\psi(x, t)$.

(c) Generalize this result to an arbitrary field characterized by a potential $V(x)$.

B4. A particle of mass m is incident on a potential barrier(you may assume this is a one dimensional problem)
$V(x)=0$, $x<0$
$V(x)=U$, $x>0$.
Solve the time-independent Schroedinger equation assuming that particle's energy $E<U$, find the particle's wavefunction in the whole space and prove that
(a) the reflection coefficient is 1.
(b) The particle's wavefunction can always be chosen real.
(c) Evaluate the penetration depth for E=10 MeV protons if U=20 MeV.

Physical constants

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$\cos(ix) = \cosh(x)$
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For small x:

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$$\cos x\approx 1-\tfrac{1}{2}x^2$$
$$\tan x\approx x+\tfrac{1}{3}x^3$$

QUANTUM MECHANICS

$E^2=p^2c^2+m^2c^4$

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x,L_y]=i\hbar L_z \quad et\ cycl.$

Ladder operators:
$$L_+\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell,m+1\rangle$$
$$L_-\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell,m-1\rangle$$

Gyromagnetic ratio for electron (SI units) = e/m

In the Heisenberg picture, for a given operator Q,

$$\frac{d<Q>}{dt}=\frac{i}{\hbar}[H,Q]$$

Pauli spin matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

For spin ½ system, the eigenvectors are

$$|+z> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-z> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+x> = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-x> = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad |+y> = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |-y> = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Barrier penetration in the WKB approximation: $P = \exp\left\{-\frac{2}{\hbar}\int\sqrt{2m\left[V(x)-E\right]}dx\right\}$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z) = \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z) = \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z) = \mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z) = \mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

Hydrogen atom: $E_n = -\frac{Ry}{n^2}$, $Ry = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$

Radial functions for the hydrogen atom $R_{nl}(r)$:

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\exp\left(-\frac{r}{a_0}\right),$$

$$R_{20}(r) = \frac{2}{(2a_0)^{3/2}}\left[1-\frac{r}{2a_0}\right]\exp\left(-\frac{r}{2a_0}\right),$$

$$R_{21}(r) = \frac{r}{24^{1/2}a_0^{5/2}}\exp\left(-\frac{r}{2a_0}\right).$$

Stationary states of harmonic oscillator for $n = 0$ and $n = 1$:

$$\varphi_0(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\frac{\alpha^2x^2}{2}},$$

$$\varphi_1(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} 2ax\,\mathrm{e}^{-\frac{\alpha^2 x^2}{2}},$$

where $\alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$.

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int_0^\infty \frac{1}{1+bx^2}\,dx = \frac{\pi}{2b^{1/2}}$$

$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$$

$$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}\,dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan\left(\dfrac{x}{b}\right)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) =$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$

$$\int x^4 e^{-x}\,dx = -e^{-x}\left(x^4+4x^3+12x^2+24x+24\right)$$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

$$\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_0^\infty x^5 e^{-x^2}\,dx = \frac{1}{a^3}$$

$$\int_0^\infty x^6 e^{-x^2}\,dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$$