Thermo A1

$p = nkT$, $\quad \frac{m v_{rms}^2}{2} = \frac{3}{2} kT$

(a)

$$v_{rms}^2 = \frac{3}{m} kT = \frac{3}{m} \frac{p}{n} = \frac{3}{3.32 \times 10^{-27} kg} \frac{1.013 \times 10^5 \frac{N}{m^2}}{2 \times 10^{25} m^{-3}}$$

$$= 0.458 \times 10^7 \frac{m^2}{s^2}$$

$$v_{rms} = 2.14 \times 10^3 \frac{m}{s}$$

(b) to increase v_{rms} by 2, pressure should be increased by 4 to 4 atm.

Thermo B4

(a) $$f(v)\,dv = \frac{4}{\pi^{1/2}}\left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\,dv$$

$$E = \frac{mv^2}{2} \qquad dE = mv\,dv$$

$$f(E)\,dE = \frac{4}{\pi^{1/2}}\left(\frac{m}{2kT}\right)^{3/2}\left(\frac{2E}{m}\right)^{1/2}\frac{1}{m}e^{-E/kT}\,dE$$

$$= \frac{2}{\pi^{1/2}(kT)^{3/2}} E^{1/2} e^{-E/kT}\,dE$$

(a) at $T = 300\,K$ $\quad kT = 0.02585\,eV$

therefore $E_0 = 0.001\,eV \ll kT$ and exponential can be replaced by 1

and $$p = \frac{2}{\pi^{1/2}(kT)^{3/2}}\int_0^{E_0} E^{1/2} e^{-E/kT}\,dE \approx \frac{2}{\pi^{1/2}(kT)^{3/2}}\cdot\frac{2}{3}E_0^{3/2} = \frac{4}{3\pi^{1/2}}\left(\frac{E_0}{kT}\right)^{3/2}$$

$$p \approx \frac{4}{3\pi^{1/2}}\left(\frac{0.001}{0.02585}\right)^{3/2} \approx 0.0057$$

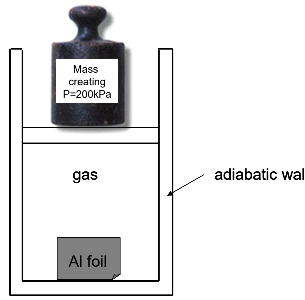
(b) $E_0 = 0.2\,eV \gg kT$, therefore $E^{1/2}$ can be replaced by $E_0^{1/2}$

$$p = \frac{2}{\pi^{1/2}(kT)^{3/2}}\int_{E_0}^{\infty} E^{1/2} e^{-E/kT}\,dE \approx \frac{2E_0^{1/2}}{\pi^{1/2}(kT)^{3/2}}kT\,e^{-E_0/kT}$$

$$= \frac{2}{\pi^{1/2}}\left(\frac{E_0}{kT}\right)^{1/2} e^{-E_0/kT} \approx 0.00137$$

Thermal solutions. Author: Binek

B1. Ten grams (10.0 g) of aluminium foil at 30^0 C (303.15 K) and 0.50 moles of an ideal gas at 15^0 C (288.15 K) are placed in a container whose volume changes to maintain the pressure at 200 kPa. The foil and the gas come to equilibrium with negligible heat transfer to or from the container. The molar specific heat of the gas is c_p=3.5 R, where R=8.314 J/(mol K) is the universal gas constant and the specific heat capacity of Al is given by $c_P^M = 0.904 \text{ kJ}/(\text{kg K})$.



a) Find the final temperature of the aluminium foil and the gas.

Heat flowing out of the Al foil into the gas where it is used to increase the internal energy and to do work.

$$Q^{Al} + Q^{gas} = 0$$

The heat is exchanged at constant pressure. Hence

$$Mc_P^M (T_f - T_{Al}) + nc_p (T_f - T_{gas}) = 0$$

Solving the equation with respect to T_f yields:

$$T_f = \frac{Mc_P^M T_{Al} + nc_P T_{gas}}{Mc_P^M + nc_P}$$

$$T_f = \frac{0.01kg\ 904J/kgK\ 303.15K + 0.5\ mol \times 3.5 \times 8.314J/mol\ K\ 288.15K}{0.01kg\ 904J/kgK + 0.5\ mol \times 3.5 \times 8.314J/mol\ K} = 293.9K = 20.75^o C$$

b) Find the work done by the gas

$$\Delta U = Q - W$$

Rearranging yields

$$W = Q - \Delta U$$

With $Q = nc_P(T_f - T_{gas})$ and $\Delta U = nc_V(T_f - T_{gas})$ we obtain

$$W = nc_P(T_f - T_{gas}) - nc_V(T_f - T_{gas})$$

Using $c_P - c_V = R$ yields

$$W = nR(T_f - T_{gas})$$

$$T_f - T_{gas} = 293.9K - 288.15K = 5.75K$$

$$W = 0.5\ mol \times \frac{8.314J}{mol\ K}\ 5.75K = 23.9\ J$$

A2. Consider an ideal gas in a container of adjustable volume, *V*, which in addition allows for control of the temperature, *T*. If you want to achieve that the pressure, *P*, increases linearly with the volume according to *P*=*AV* with *A*=const you have to increase the temperature while the volume increases.

a) Find the functional form $T=T(V)$ which allows to realize $P=A\ V$.

$$P = \frac{nRT}{V} = A\ V$$

Hence

$$T(V) = \frac{AV^2}{nR}$$

b) The temperature of the container is increased to T_f while the pressure changes according to $P=AV$ and the volume increases from $V_i=V_0$ to $V_f=2V_0$. Find the work done by the gas in terms of n, R and T_f.

Hint: In case you could not solve a) express the work in terms of A and V_0 for partial credit.

$$W = \int_{V_i}^{V_f} PdV = \frac{1}{2}A\left(V_f^2 - V_i^2\right) = \frac{3}{2}AV_0^2$$

Using

$$T(V) = \frac{AV^2}{nR}$$

And thus

$$T_f = \frac{AV_f^2}{nR} = \frac{4AV_0^2}{nR}$$

We obtain

$$AV_0^2 = \frac{nR}{4}T_f$$

And therefore

$$W = \frac{3}{2}AV_0^2 = \frac{3}{8}nRT_f$$

B2. A certain volume of water with constant heat capacity C_P is initially at T_i. It is brought into contact with a heat reservoir at temperature T_r.

a) What is, ΔS_{total}, the entropy change of the entire system (water and reservoir) when the water reaches the temperature of the heat reservoir? Assume that in good approximation the volume of the water doesn't change on temperature change.
Express the answer in terms of C_P, T_i, and T_r .

Hint: Think about sign of the heat flow from or into the reservoir.

$$\Delta S_{water} = C_P \int_{T_i}^{T_r} \frac{dT}{T} = C_P \ln \frac{T_r}{T_i}$$

$$\Delta S_{reservoir} = \frac{Q}{T_r} = \frac{-C_P(T_r - T_i)}{T_r} = -C_P(1 - \frac{T_i}{T_r})$$

b) Show that $\Delta \mathrm{S}_{\mathrm{total}}(\frac{T_r}{T_i}) \geq 0$ for all $\frac{T_r}{T_i} > 0$ by discussing and sketching the function $\Delta \mathrm{S}_{\mathrm{total}}(\frac{T_r}{T_i})$.

$$\Delta \mathrm{S}_{\mathrm{total}} = \Delta S_{water} + \Delta S_{reservoir} = C_P \ln \frac{T_r}{T_i} - C_P(1 - \frac{T_i}{T_r})$$

The total entropy increases because the process is irreversible.

$\Delta \mathrm{S}_{\mathrm{total}}\left(\frac{T_r}{T_i}\right) = C_P \ln \frac{T_r}{T_i} - C_P(1 - \frac{T_i}{T_r}) \geq 0$ all $\frac{T_r}{T_i} > 0$.

To see that we discuss the function $\Delta \mathrm{S}_{\mathrm{total}}(x) = C_P \ln x - C_P(1 - \frac{1}{x})$

Finding extremum:

$$\frac{d\Delta \mathrm{S}_{\mathrm{total}}(x)}{dx} = C_P \left[\frac{1}{x} - \frac{1}{x^2}\right] = 0$$

Extremum at $x = 1$

The extremum is a minimum as can be seen from

$$\frac{d^2\Delta \mathrm{S}_{\mathrm{total}}(x)}{dx^2} = C_P \left[-\frac{1}{x^2} + \frac{2}{x^3}\right]$$

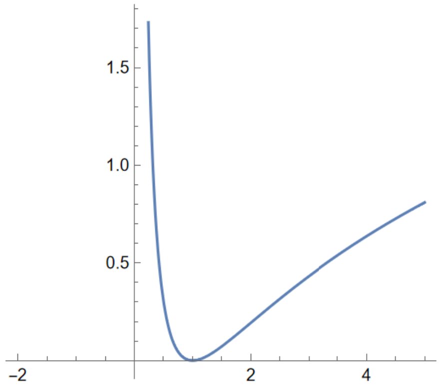
$$\frac{d^2\Delta \mathrm{S}_{\mathrm{total}}(x = 1)}{dx^2} = C_P > 0$$

For $x \gg 1$, $\Delta \mathrm{S}_{\mathrm{total}}(x) \to \ln x$ and for $x \to 0$, $\Delta \mathrm{S}_{\mathrm{total}}(x) \to \frac{1}{x}$ this can be seen, e.g., by expanding $\ln x$ around $x = 1$.

With $\frac{d}{dx} \ln x = \frac{1}{x}$ one obtains $\ln x = (x - 1) + \cdots$

Which yields $\Delta \mathrm{S}_{\mathrm{total}}(x) \approx C_P\,(x - 2 + \frac{1}{x})$ in the vicinity of $x = 1$.

This part is not necessary to get full credit. All they need to show is that the entropy change is zero for $\frac{T_r}{T_i} = 1$ and larger than zero everywhere else which is evident once showing that $\frac{T_r}{T_i} = 1$ is a minimum and the only minimum.



B3. A cyclic equilibrium process in *n* moles of an ideal gas with $c_V = 2.5R$ is formed of three sub-processes:

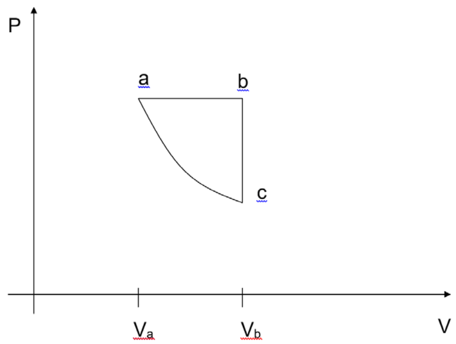
a $\rightarrow$ b is a constant pressure doubling of the volume;

b $\rightarrow$ c is at constant volume with decreasing pressure;

c $\rightarrow$ a is adiabatic.

Assume that $V_b = V_c = 2V_a$

(a) Sketch the process on a *PV* diagram.



(b) Find the heat absorbed in part a $\rightarrow$ b in terms of T_a, *n*, *R* and a number.
a->b is a constant pressure process, hence $Q = n\,c_P \Delta T$. Since we have an ideal gas with c_V=2.5R, c_p is given by $c_P = c_V + R = 3.5R$.

Since the pressure is constant we calculate the temperature change from $\frac{nRT_a}{V_a} = \frac{nRT_b}{V_b}$. With V_b=2V_a we obtain $\frac{T_a}{V_a} = \frac{T_b}{2V_a}$ and hence, T_b=2T_a.

Therefore the absorbed heat reads $Q = n\, c_P \Delta T = n\, 3.5R\, T_a$

(c) Find the heat rejected in part b$\rightarrow$ c in terms of T_a , n, R and a number.

b->c is a constant volume process, hence $Q = n\, c_V \Delta T$. The temperature in point b) is $T_b=2T_a$. The temperature in point c) is determined with the help of the equation for the adiabatic change between c) and a). From $PV^\gamma = \text{const.}$ we obtain with the help of the ideal gas equation of state: $T_c V_c^{\gamma-1} = T_a V_a^{\gamma-1}$. With $V_c=V_b=2V_a$ we obtain

$$T_c = T_a \left(\frac{1}{2}\right)^{\gamma-1} \quad \text{where} \quad \gamma = \frac{c_P}{c_V} = \frac{3.5}{2.5} = 1.4$$

This yields $Q = n\, c_V \Delta T = n\, 2.5\, RT_a (0.5^{0.4} - 2) = -3.105nRT_a$

(d) Find the energy efficiency of the cycle. Give a numerical answer.

The energy efficiency is defined according to $\eta = 1 - \frac{|Q_{out}|}{|Q_{in}|}$. This yields for our example:

$$\eta = 1 - \frac{3.105}{3.5} = 0.114 = 11.4\%$$

A4. Air (approximated as an ideal gas with $c_V = 2.5R$) initially at 293K ($20°C$) is adiabatically compressed

(a) Find the final temperature when the compression ratio V_f/V_0 is $1/10$ (as is typical in gasoline engines).

We use the formula derived in the above problem $T_o V_o^{\gamma-1} = T_f V_f^{\gamma-1}$. We obtain:

$$T_f = T_o \frac{V_o^{\gamma-1}}{V_f^{\gamma-1}} = 293(10)^{0.4}\,\text{K} = 736\text{K}$$

(b) Find the final temperature when the compression ratio V_f/V_0 is $1/20$ (as is typical in diesel engines).

$$T_f = T_o \frac{V_o^{\gamma-1}}{V_f^{\gamma-1}} = 293(20)^{0.4}\,\text{K} = 971\text{K}$$

A3. Of the 20 faces in 10 coins, 13 are tails: 5 tails on the 5 normal coins plus 8 tails on the 4 two-tail coins. Of these 13 tails, there are 5 tails on normal coins.

Hence, the probability of drawing a tail on a normal coin *under the condition that a tail was drawn* is

5 out of 13 = 5/13 = 38.4%

Sample Prelim Questions

May 10, 2022

The first four problem are at the physics 211H level. The following four problems are at a physics 311 level.

Problem 1

A1

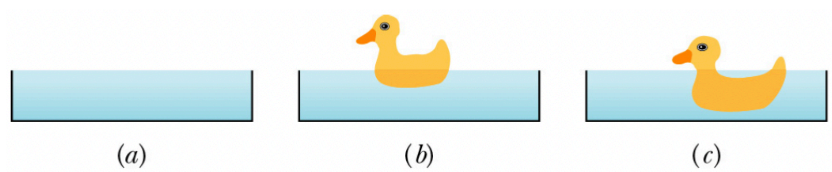


Figure 1: Problem 1

Three identical open-top containers are filled to the brim with water. Toy ducks float in two of the containers, as shown. Rank the containers plus contents according to their weight, e.g. $(a) > (b) = (c)$. Explain your reasoning.

Solution:
The buoyant force is given by the weight of the volume of water displaced, which is equal to the weight of the floating object displacing the fluid. Thus, $(a) = (b) = (c)$.

Problem 2

B1

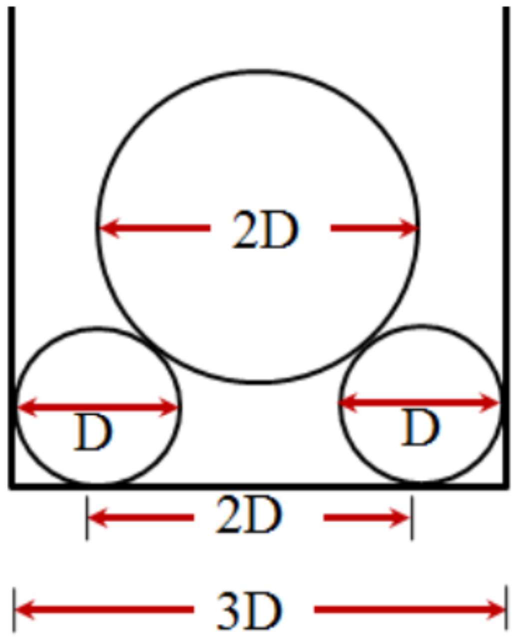


Figure 2: Problem 2

Three pipes with smooth walls rest in an open box (width $3D$) with a horizontal bottom and vertical walls. Two of the pipes have diameter D and weight W_1 and sit on the bottom of the box with centers separated by distance $2D$. The third pipe has diameter $2D$ and weight W_2 and rests on the other pipes, as shown. Calculate the force on each of the vertical walls.

Solution:

This is a static equilibrium problem. Let N_D be the normal force of the floor of the container acting on a pipe of diameter D, N_{2D} be the normal force between pipes of diameter D and $2D$, and F_V be the force on each of the vertical walls. Let θ be the angle that a line connecting the centers of the pipes of diameters D and $2D$ makes with the horizontal passing through the centers of the pipes of diameter D. Applying Newton's second law to the pipe of diameter $2D$ gives

$$\sum F_{2D,x} = 0 = N_{2D} \cos\theta - N_{2D} \cos\theta$$
$$\sum F_{2D,y} = 0 = 2N_{2D} \sin\theta - W_2 \tag{1}$$

Performing the same exercise for the left pipe with diameter D yields

$$\sum F_{D,x} = 0 \quad = \quad F_V - N_{2D} \cos\theta$$
$$\sum F_{D,y} = 0 = N_D - N_{2D} \sin\theta - W1$$

The equations for the right pipe of diameter D are symmetric. From above we obtain $F_V = N_{2D} \cos\theta$ and $N_{2D} = \frac{W_{2D}}{2\sin\theta}$ and thus $F_V = \frac{W_2}{2\tan\theta}$. It remains to determine $\tan\theta$ which can be found with some geometry. The length of a line connecting the centers of the pipes of diameters D and $2D$ is $\frac{3D}{2}$, and the horizontal distance between the centers of the two pipes is D. The vertical distance between the centers is thus of $\frac{\sqrt{5}}{2} D$ and $\tan\theta = \frac{\sqrt{5}}{2}$. The force on each the vertical walls is then $F_V = \frac{W_2}{\sqrt{5}}$. W2/sqrt{5}

Problem 3

A3

Determine the wavelengths of the three lowest-frequency tones produced by a pipe of length L that is open at both ends.

Solution: A pipe open at both ends has boundary conditions that are pressure nodes (or displacement antinodes). Thus, the standing pressure in the pipe is of the form $p(x,t) = A \sin kx \sin \omega t$. Applying boundary conditions,

$$\begin{aligned} p(0,t) &= 0 \\ p(L,t) = 0 &= A \sin kL \sin \omega t \end{aligned}$$

From the second equation above, it follows that $\sin kL = 0$ or $k_n = n\frac{\pi}{L}$. The tonal wavelengths are given by $\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$. Thus, the wavelength of the three lowest-frequency (longest wavelength) tones is given $\lambda = 2L, L, \frac{2L}{3}$.

Problem 4

A4

Consider a ball launched from level ground at a fixed angle θ and a speed v_0. The objective is to shoot the ball through a window a distance L away that is at a height h. Find an expression for the speed v_0 required to shoot the ball through the window. Your answer depend only on the parameters defined here and the acceleration due to gravity, g.

Solution: The components of the acceleration of the ball are $a_x = 0, a_y = -g$, where the negative sign indicates that the acceleration due to gravity acts downwards and g has a magnitude of 9.8 m/s^2. The kinematics are then described by the relations

$$\begin{aligned} v_x(t) &= v_0 \cos\theta \\ x(t) &= v_0 \cos\theta t \\ v_y(t) &= v_0 \sin\theta - gt \\ y(t) &= v_0 \sin\theta t - \frac{1}{2} g t^2 \end{aligned}$$

The ball passes through the window after traveling a distance L along the x-direction, which happens at a time $T = \frac{L}{v_0 \cos\theta}$. Substituting the expression for T into that for $y(t)$ yields

$$y(T) = v_0 \sin\theta \left(\frac{L}{v_0 \cos\theta} \right) - \frac{g}{2} \left(\frac{L}{v_0 \cos\theta} \right)^2$$

Setting $y(T) = h$ and solving for v_0 yields

$$v_0 = \frac{L}{\cos\theta} \sqrt{\frac{g}{2\,(L \tan\theta - h)}}$$

Problem 5

A2

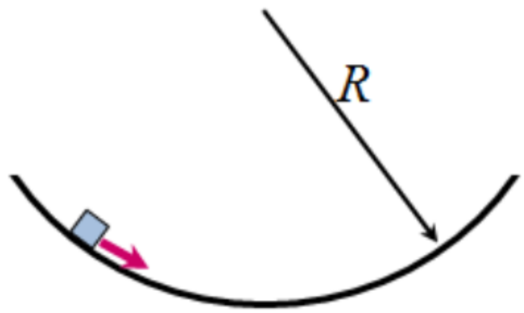


Figure 3: Problem 5

A particle slides frictionlessly inside a spherical surface of radius R, as shown. Show that the motion is simple harmonic for small displacements and find the period of this motion.

Solution: Let θ be the angular coordinate of mass m measured with respect to a vertical line that passes through the center of the spherical surface. Let $x_\perp$ be the coordinate along a direction tangent to the spherical surface at the point of the mass m. The forces acting on m can be written in terms of components that lie along the radius of the spherical surface and perpendicular to it, yielding

$$\begin{aligned} \sum F_r = 0 &= N - mg\cos\theta \\ \sum F_\perp = ma_\perp &= -mg\sin\theta \end{aligned}$$

For small displacements, $\sin\theta \approx \frac{x_\perp}{R}$ and the resulting equation of motion along a direction tangent to the spherical surface is $\ddot{x}_\perp + \omega^2 x_\perp = 0$, where $\omega^2 = \frac{g}{R}$. Thus, the motion is simple harmonic with period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$.

Problem 8 B3

A uniform density solid cylinder of mass M and radius R is free to rotate about an axis, which is horizontal. The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Part of a long cable of negligible mass is wound around the cylinder with the remainder of the cable hanging vertically. A massless spring, with spring constant k, is attached to the end of the cable, and a mass m is attached to the end of the spring. Determine the Lagrangian using a suitable choice of generalized coordinates and the resulting equations of motion. Find the conjugate momenta and Hamiltonian.

Solution: Let coordinate θ describe the angular displacement of the cylinder and coordinate y describe the length of string from the cylinder to the spring. Also, take the rest length of the spring to be ℓ and the stretch of the spring by s, such that the total length of the spring is given by $\ell + s$. The kinetic and potential energies in terms of the coordinates θ, y, s are given by

$$
\begin{aligned}
T &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\left(\dot{y}^2 + \dot{s}^2\right) = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{y}^2 + \frac{1}{2}m\left(\dot{s}^2 + 2\dot{y}\dot{s}\right) \\
U &= mg(y + s + \ell) + \frac{1}{2}ks^2,
\end{aligned}
$$

where in the expression for the kinetic energy the constraint $R\theta - y = 0$ is imposed. The resulting Lagrangian is

$$
L = T - U = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{y}^2 + \frac{1}{2}m\left(\dot{s}^2 + 2\dot{y}\dot{s}\right) - mg(y + s + \ell) - \frac{1}{2}ks^2.
$$

The equations of motion determined using Euler-Lagrange are

$$
\begin{aligned}
\left(m + \frac{I}{R^2}\right)\ddot{y} + m\ddot{s} + mg = 0 \\
\ddot{y} + \ddot{s} + g + \omega^2 s = 0,
\end{aligned}
$$

where $\omega^2 = \frac{k}{m}$. The conjugate momenta can be determined from the Lagrangian as

$$p_y = \frac{\partial L}{\partial \dot{y}} = \left(m + \frac{I}{R^2}\right)\dot{y} + m\dot{s}$$
$$p_s = \frac{\partial L}{\partial \dot{s}} = m(\dot{s} + \dot{y})$$

and the Hamiltonian H is

$$H = \sum_k p_k \dot{q}_k - L = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{y}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{y}\dot{s} + mg(y + s + \ell) + \frac{1}{2}ks^2$$

The Hamiltonian is the total energy $H = T + U$. As the Lagrangian has no explicit time dependence, i.e. $\frac{\partial L}{\partial t} = 0$, H is a conserved quantity.

CM B2

(a) $$\frac{dm}{dt} = kA = k\pi r^2$$

also $$\frac{dm}{dt} = \frac{d}{dt}\left(\rho \cdot \frac{4}{3}\pi r^3\right) = 4\pi\rho r^2 \dot{r}$$

therefore $$k\pi r^2 = 4\pi\rho r^2 \dot{r}$$

$$\dot{r} = \frac{k}{4\rho} \longrightarrow r = r_0 + \frac{k}{4\rho} t$$

(b)

Newton 2nd law

$$\frac{d}{dt}(mv) = mg$$

$$v\frac{dm}{dt} + m\frac{dv}{dt} = mg \qquad (1)$$

from (a) $$\frac{dm}{dt} = k\pi\left(r_0 + \frac{k}{4\rho}t\right)^2$$

$$m(t) - m(0) = \int_0^t k\pi\left(r_0 + \frac{k}{4\rho}t\right)^2 dt = k\pi \frac{4\rho}{k} \cdot \frac{1}{3}\left(r_0 + \frac{k}{4\rho}t\right)^3\Big|_0^t$$

$$= \frac{4}{3}\pi\rho\left(r_0 + \frac{k}{4\rho}t\right)^3\Big|_0^t$$

with $m(0) = \frac{4}{3}\pi\rho r_0^3$

we get $$m(t) = \frac{4}{3}\pi\rho\left(r_0 + \frac{k}{4\rho}t\right)^3$$

$$\frac{1}{m}\frac{dm}{dt} = \frac{3}{4}\frac{k}{\rho}\frac{1}{\left(r_0 + \frac{k}{4\rho}t\right)},$$ and from Eq. (1)

$$\boxed{\frac{3k}{4\rho\left(r_0 + \frac{k}{4\rho}t\right)} v + \frac{dv}{dt} = g}$$

CM

B4.

$$m\ddot{x} = F_0 \cos\omega t$$

$$\dot{x} = \frac{F_0}{m\omega}\sin\omega t + v_0$$

(a) $$x = \frac{F_0}{m\omega^2}\cos\omega t + v_0 t + C$$

$$x(0) = 0 \rightarrow \quad C = -\frac{F_0}{m\omega^2}$$

$$x = \frac{F_0}{m\omega^2}(\cos\omega t - 1) + v_0 t$$

(b) $$\dot{x}^2 = \left(\frac{F_0}{m\omega}\right)^2 \sin^2\omega t + \frac{2F_0 v_0}{m\omega}\sin\omega t + v_0^2$$

$$\langle \sin^2\omega t \rangle = \frac{1}{2} \quad \langle \sin\omega t \rangle = 0$$

therefore $\langle \dot{x}^2 \rangle = \frac{1}{2}\left(\frac{F_0}{m\omega}\right)^2 + v_0^2$

$$\langle K \rangle = \frac{m\langle \dot{x}^2 \rangle}{2} = \frac{F_0^2}{4m\omega^2} + \frac{m v_0^2}{2}$$