Section 4: Electromagnetic Waves 2

Frequency dependence and dielectric constant

We now consider frequency dependence of electromagnetic waves propagating in a dielectric medium. As before we suppose that the medium is not magnetized, so that we can set $\mathbf{M} = 0$ in our equations. We further suppose that the waves are propagating in the absence of free charges and currents. Maxwell's equations therefore are reduced to

\[
\nabla \cdot \mathbf{D} = 0 \quad \text{(4.1)}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{(4.2)}
\]

\[
\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad \text{(4.3)}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} \quad \text{(4.4)}
\]

where $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$.

We want to form plane-wave solutions to these equations, and examine how the waves interact with the medium. Here we assume that the medium is uniform, isotropic, and unbounded. To begin we will look for monochromatic waves oscillating with an angular frequency $\omega$. We therefore set

\[
\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} \quad \text{(4.5)}
\]

with similar expressions for $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{P}$. Note that only the real parts of these complex fields have physical meaning.

The restriction to monochromatic waves is not too severe. Because the field equations are linear (assuming, as we shall verify below, that the relation between $\mathbf{D}$ and $\mathbf{E}$ is itself linear), solutions with different frequencies can be added to form wave-packet solutions. For example, we can express the electric field as

\[
\mathbf{E}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(\omega, \mathbf{r}) e^{-i\omega t} d\omega \quad \text{(4.6)}
\]

in terms of the frequency-domain field $\mathbf{E}(\omega, \mathbf{r})$; this is a Fourier representation of the electric field. The inverse transformation is

\[
\mathbf{E}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{i\omega t} dt \quad \text{(4.7)}
\]

and frequency-domain fields $\mathbf{D}(\omega, \mathbf{r})$, $\mathbf{B}(\omega, \mathbf{r})$, and $\mathbf{P}(\omega, \mathbf{r})$ can be defined a similar way.

The frequency-domain fields satisfy the equations

\[
\nabla \cdot \mathbf{D} = 0 \quad \text{(4.8)}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{(4.9)}
\]

\[
\nabla \times \mathbf{E} = -i\omega \mathbf{B} = 0 \quad \text{(4.10)}
\]

\[
\nabla \times \mathbf{B} + i\omega \mu_0 \mathbf{D} = 0 \quad \text{(4.11)}
\]

and we have the relation
These are the sourceless Maxwell’s equations in the frequency domain. They can only be solved once an explicit relationship (called a constitutive relation) is introduced between the polarization $\mathbf{P}$ and the electric field $\mathbf{E}$. The constitutive relation describes the medium’s response to an applied electric field, and its derivation is based on complicated molecular processes. While quantum mechanics is required for a proper treatment, in the next section we shall consider a simple classical model that nevertheless captures the essential features. For simplification of the notation we will denote below the Fourier components of all the fields with no tilde.

**Dielectric constant**

We suppose that the medium is sufficiently dilute that the electric field felt by any given molecule is just the applied field $\mathbf{E}$ itself, the field exerted by other molecules being negligible. We assume that in the absence of an applied field, the molecular dipole moment is zero, at least on average. When an electric field is applied, however, some of the molecular charges (the electrons) move away from their equilibrium positions, and a dipole moment develops.

The displacement of charge bound charge $q_b$ from its equilibrium position is denoted $\xi_b(t)$. This charge undergoes a motion that is governed by the applied electric field (which tends to drive $\xi_b$ away from zero) and the intra-molecular forces (which tend to drive the charge back to its equilibrium position). We model these forces as harmonic forces, all sharing the same natural frequency $\omega_0$. What we have, therefore, is a system of simple harmonic oscillators of natural frequency $\omega_0$ that are driven at a frequency $\omega$ by an applied force. Since the motion of the charges will produce electromagnetic radiation, energy will gradually be removed from the oscillators (to be carried off by the radiation), and the oscillations will be damped. In addition, the interaction between molecules and other charges in the system will also result in the damping of the oscillations. We incorporate this effect through a phenomenological damping parameter $\gamma > 0$. The equations of motion for the charge $q_b$ are therefore

$$m_b \left( \frac{d^2 \xi_b}{dt^2} + \gamma \frac{d \xi_b}{dt} + \omega_0^2 \xi_b \right) = -q_b \mathbf{E}(\mathbf{r}_n, \omega) e^{-i\omega t}. \quad (4.13)$$

Notice that the applied electric field $\mathbf{E}(\mathbf{r}_n, \omega)$, which is macroscopic and varies slowly over intra-molecular distances, is evaluated at the molecule’s center of mass $\mathbf{r}_n$. To solve the equations of motion in the steady-state regime we write $\xi_b(t) = \bar{\xi}_b(\omega) e^{-i\omega t}$ and we substitute this into Eq.(4.13). This yields

$$\bar{\xi}_b(\omega) = \frac{q_b / m_b}{\omega_0^2 - \omega^2 - i\gamma \omega} \mathbf{E}(\mathbf{r}_n, \omega). \quad (4.14)$$

where we again eliminated the tilde from the notation of the Fourier component.

Since the dipole moment vanishes at equilibrium, the molecule’s dipole moment is given by

$$\mathbf{p}_n = \sum_b q_b (\mathbf{r}_b + \bar{\xi}_b) = \sum_b q_b \bar{\xi}_b. \quad (4.15)$$

where $\mathbf{r}_b$ is the equilibrium position of charge $q_b$ with respect to the center of mass. Substituting Eq.(4.14) then gives

$$\mathbf{p}_n = \sum_b \frac{q_b^2 / m_b}{\omega_0^2 - \omega^2 - i\gamma \omega} \mathbf{E}(\mathbf{r}_n, \omega). \quad (4.16)$$
Since all the contributing charges \( q_b \) are electrons, we can assume that \( \sum_b q_b^2/m_b = Ze^2/m \), where \( Z \) is the number of electrons per molecule, \( e \) is the electronic charge, and \( m \) the electron’s mass. We therefore arrive at

\[
p_a = \frac{Ze^2}{m} \frac{1}{\omega_p^2 - \omega^2 - i\gamma_\omega} E(r_a, \omega) .
\]  

(4.17)

for the molecular dipole moment in the frequency domain. Assuming that medium has \( N \) molecules per macroscopic unit volume we find that the polarization vector

\[
P(r, \omega) = \frac{Ze^2 N}{m} \frac{1}{\omega_p^2 - \omega^2 - i\gamma_\omega} E(r, \omega) .
\]  

(4.18)

Now using the relationship \( D = \varepsilon_0 E + P \) we find

\[
D(r, \omega) = \varepsilon_0 E + P = \varepsilon_0 \left( 1 + \frac{Ze^2 N}{\varepsilon_{jm} \omega_p} \frac{1}{\omega_p^2 - \omega^2 - i\gamma_\omega} \right) E(r, \omega) .
\]  

(4.19)

or

\[
D(r, \omega) = \varepsilon(\omega) E(r, \omega) .
\]  

(4.20)

where we introduced the (complex and frequency-dependent) dielectric constant such that

\[
\varepsilon(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_p^2 - \omega^2 - i\gamma_\omega} .
\]  

(4.21)

where we defined the medium’s “plasma frequency”

\[
\omega_p = \left( \frac{Ze^2 N}{\varepsilon_{jm}} \right)^{1/2} .
\]  

(4.22)

We note though that the term “plasma frequency” have only meaning for free electrons. For bound electrons \( \omega_p \) should be regarded as some constant.

Equation (4.21) is the result of a simplistic classical model. But a proper quantum-mechanical treatment would produce a similar result, except for the fact that the electrons actually oscillate with a discrete spectrum of natural frequencies \( \omega_j \) and damping coefficients \( \gamma_j \). A more realistic expression for the dielectric constant is then

\[
\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} .
\]  

(4.23)

where \( f_j \) is the fraction of electrons that share the same natural frequency \( \omega_j \). We will continue to deal with the simple expression of Eq.(4.21). It is good to keep in mind, however, that the model is a bit crude.

Fig. 4.1 shows the real and imaginary parts of the dielectric constant. It is seen that that for most frequencies, when \( \omega \) is not close to \( \omega_p \), Re(\( \omega \) \( \omega \)) increases with increasing \( \omega \), and Im(\( \omega \) \( \omega \)) is very small; this typical behavior is associated with normal dispersion. When \( \omega \) is very close to \( \omega_p \), however, the electrons are driven at almost their natural frequency, and resonance occurs. Near resonance, then, Re(\( \omega \) \( \omega \)) decreases with increasing \( \omega \), and Im(\( \omega \) \( \omega \)) is no longer small; this behavior is associated with anomalous dispersion. In realistic models the resonances occur at a number of frequencies \( \omega_j \). Normal
dispersion is seen to occur everywhere except the neighborhood of a resonance frequency. At this
frequency the imaginary part of $\varepsilon$ is appreciable. Since a positive Im$\varepsilon$ represents dissipation of energy
from the EM wave into the medium, the regions where Im$\varepsilon$ is large is called resonant absorption.

![Fig. 4.1. Real and imaginary parts of the dielectric constant, as functions of $\omega/\omega_0$. The real part of
the dielectric constant goes through 1 at resonance. The imaginary part of the dielectric constant
achieves its maximum at resonance.](image)

At low frequencies the dielectric constant behaves as $\varepsilon_r(\omega)\approx1+\left(\frac{\omega_p}{\omega_0}\right)^2$, and is therefore a constant
slightly larger than unity. At high frequencies it behaves as $\varepsilon_r(\omega)\approx1-\left(\frac{\omega_p}{\omega}\right)^2$, and therefore
approaches unity from below.

**Monochromatic plane waves**

After making the substitution of Eq.(4.20), the frequency-domain Maxwell equations become

$$\nabla \cdot E = 0$$  \hspace{1cm} (4.24)

$$\nabla \cdot B = 0$$  \hspace{1cm} (4.25)

$$\nabla \times E - i\omega B = 0$$  \hspace{1cm} (4.26)

$$\nabla \times B + i\omega \varepsilon(\omega)\mu_0 E = 0,$$  \hspace{1cm} (4.27)

Similar to what we have done before, it is easy to show that as a consequence of Eqs.(4.24)-(4.27), both $E$
and $B$ satisfy a frequency-domain wave equation. For example, taking the curl of Eq.(4.26) gives

$$\nabla \times (\nabla \times E) - i\omega \nabla \times B = 0$$  \hspace{1cm} (4.28)

substituting Eq. (4.27) and making use of the vectorial identity $\nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E$ puts this in
the form

$$(\nabla^2 + k^2)E = 0,$$  \hspace{1cm} (4.29)

where we have introduced a complex wave number $k(\omega)$

$$k^2(\omega) \equiv \frac{\omega^2 \varepsilon(\omega)}{c^2 \varepsilon_0}$$  \hspace{1cm} (4.30)
and we recall that \( c \equiv (\varepsilon_0 \mu_0)^{-1/2} \) is the speed of light in vacuum. We can also introduce a complex index of refraction

\[
 n^2(\omega) \equiv \frac{\varepsilon(\omega)}{\varepsilon_0}
 \]

(4.31)

Similar manipulations return

\[
 (\nabla^2 + k^2) \mathbf{B} = 0,
 \]

(4.32)

the statement that the magnetic field also satisfies a wave equation.

To see how monochromatic electromagnetic waves propagate in a dielectric medium suppose that the wave is a plane wave that propagates in the \( z \) direction, so that \( \mathbf{E}, \mathbf{B} \sim e^{ikz} \). This is similar to what we had before but now the wave vector \( k \) is a complex number. To identify the physical properties of this wave we substitute Eq. (4.21) into Eq. (4.30) and obtain

\[
 k^2(\omega) \equiv \frac{\omega^2}{c^2} \left( 1 + \frac{\omega_p^2}{\omega^2 - \omega^2 - i\gamma\omega} \right).
 \]

(4.33)

For simplicity we suppose that the second term within the large brackets is small. Taking the square root gives

\[
 k = \frac{\omega}{c} \left( 1 + \frac{1}{2} \frac{\omega_p^2}{\omega^2 - \omega^2 - i\gamma\omega} \right).
 \]

(4.34)

The complex number can be represented as

\[
 k = k_R + \imath k_i,
 \]

(4.35)

where

\[
 k_R = \frac{\omega}{c} \left[ 1 + \frac{1}{2} \frac{\omega_p^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \right],
 \]

(4.36)

is the real part of the complex wave number, while

\[
 k_i = \frac{1}{2} \frac{\omega}{c} \frac{\omega_p \gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}.
 \]

(4.37)

is its imaginary part. The right-moving wave is therefore described by

\[
 E(z,t) = E_0 e^{ikz} e^{-\imath \omega t} = E_0 e^{-k_z z} e^{\imath k_i \omega t}.
 \]

(4.38)

From this expression we recognize that the wave is indeed traveling in the positive \( z \) direction, with a speed

\[
 v_p = \frac{\omega}{k_R(\omega)} = \frac{c}{n_k(\omega)}
 \]

(4.39)

known as the phase velocity; \( n_k \equiv \frac{k_R c}{\omega} \) is the medium’s index of refraction (a real part of the complex index of refraction). From our previous expression for \( k_R \) we obtain
and we see that \( n_r = 1 + \frac{\omega_p^2}{2\omega_0^2} > 1 \) for low frequencies, so that \( v_p < c \). On the other hand, \( n_r \approx 1 - \frac{\omega_p^2}{2\omega_0^2} < 1 \) for high frequencies, so that \( v_p > c \). The phase velocity of the wave can therefore exceed the speed of light in vacuum! This strange fact does not violate any relativistic notion, because a monochromatic wave does not carry information. To form a signal one must modulate the wave and superpose solutions with different frequencies. As we shall see, in such situations the wave always travels with a speed that does not exceed \( c \).

Another important consequence of Eq. (4.38) is that the wave is an exponentially decreasing function due to dissipation of energy. The absorption coefficient is defined by \( \alpha = 2 k_i \) and describes the attenuation of the wave intensity. It follows from eqs. (4.37) and (4.21) that \( \alpha = 2 k_i = \frac{\omega}{c} \text{Im} \varepsilon \), so that indeed the imaginary part of the dielectric constant is associated with absorption of an EM wave. The largest absorption occurs near resonances \( (\omega = \omega_j) \). At high frequencies \( (\omega \gg \omega_j) \) \( k_i \) and consequently \( \alpha \) tend to zero and therefore the medium becomes transparent.

**Low frequency behavior, electrical conductivity**

In the limit of \( \omega \rightarrow 0 \) there is qualitative difference in response of medium depending on whether the lowest resonance frequency is zero or nonzero. For insulators the lowest resonance frequency is zero or nonzero. For metals however some fraction of electrons \( f_0 \) is free in a sense of having \( \omega_0 = 0 \). The dielectric constant (4.21) is then singular at \( \omega = 0 \). If contribution from free electrons is exhibited separately, the eq. (4.21) gives

\[
\varepsilon(\omega) = \varepsilon_b(\omega) + i \frac{n_e e^2}{m \omega (\gamma_0 - i \omega)}.
\]  

(4.41)

where \( \varepsilon_b \) is the contribution from bound electrons and \( n_e = ZN f_0 \) is the concentration of free electrons.

The singular behavior can be understood if we examine the Maxwell-Ampere equation

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\]  

(4.42)

and assume that the medium obeys Ohm's law: \( \mathbf{J} = \sigma \mathbf{E} \) and has a "normal" dielectric constant \( \varepsilon_b \). With harmonic time dependence the equation becomes

\[
\nabla \times \mathbf{H} = -i\omega \left( \varepsilon_b + i \frac{\sigma}{\omega} \right) \mathbf{E}
\]  

(4.43)

If, on the other hand, we did not insert Ohm's law explicitly but attributed instead all the properties of the medium to the dielectric constant, we would identify the quantity in brackets on the right-hand side of (4.43) with \( \varepsilon(\omega) \), i.e.

\[
\varepsilon(\omega) = \varepsilon_b + i \frac{\sigma(\omega)}{\omega}
\]  

(4.44)

Comparison with (4.41) yields an expression for the conductivity
\[ \sigma(\omega) = \frac{n_e^2}{m(\gamma_0 - i\omega)} = \frac{n_e^2 e^2}{m(1 - i\omega \tau)}, \]  

(4.45)

where we introduced \( \tau \equiv 1/\gamma_0 \). This is essentially the model of Drude for the electrical conductivity. We can rewrite this expression in the form

\[ \sigma(\omega) = \frac{\sigma_0}{(1 - i\omega \tau)}, \]  

(4.46)

where \( \sigma_0 \) is the dc Drude conductivity

\[ \sigma_0 = \frac{n_e^2 e^2}{m}. \]  

(4.47)

and \( \tau = 1/\gamma_0 \) is the relaxation or scattering time. The latter can be determined empirically from experimental data on the conductivity. For copper, \( n_e = 8 \times 10^{28} \) free electrons per m\(^3\) and at normal temperatures the dc conductivity is \( \sigma_0 \approx 6 \times 10^7 \) \( \Omega^{-1}\)m\(^{-1}\). This gives \( \tau \approx 2 \times 10^{14} \) s.

The conductivity is now a complex quantity:

\[ \text{Re}\sigma = \frac{\sigma_0}{1 + \omega^2 \tau^2}; \quad \text{Im}\sigma = \frac{\sigma_0 \omega \tau}{1 + \omega^2 \tau^2}, \]  

(4.48)

The real part represents the in-phase current which produces the resistive joule heating, while the imaginary part represents the \( \pi/2 \) out-of-phase inductive current. An examination of \( \text{Re}\sigma \) and \( \text{Im}\sigma \) as functions of the frequency shows that in the low-frequency region, \( \omega \tau \ll 1 \), \( \text{Im}\sigma \ll \text{Re}\sigma \). That is, the electrons exhibit an essentially resistive character. Since \( \tau \approx 10^{14} \) s, this spans the entire familiar frequency range up to the far infrared. In the high-frequency region, \( \omega \tau \gg 1 \), that corresponds to the visible and ultraviolet regimes, \( \text{Re}\sigma \ll \text{Im}\sigma \), and the electrons display an essentially inductive character. No energy is absorbed from the field in this range, and no joule heat appears.

The dielectric function is then given by

\[ \varepsilon = \left( \varepsilon_B - \frac{\sigma_0 \tau}{1 + \omega^2 \tau^2} \right) + \frac{i\sigma_0}{\omega(1 + \omega^2 \tau^2)}. \]  

(4.49)

We consider now two frequency regions.

a) The low-frequency region \( \omega \tau \ll 1 \). In this frequency range we can neglect the real part of conductivity and find that

\[ \varepsilon = \varepsilon_B + i\frac{\sigma_0}{\omega}. \]  

(4.50)

A complex wave number \( k(\omega) \) in this region is given by

\[ k^2(\omega) = \frac{\omega^2 \varepsilon(\omega)}{c^2 \varepsilon_0} = \mu_0 \varepsilon_B \omega^2 + i\mu_0 \omega \sigma_0. \]  

(4.51)

If the frequency is sufficiently small we can neglect the real part of the dielectric constant

\[ \varepsilon \approx i\frac{\sigma_0}{\omega}, \]  

(4.52)

so that

\[ k^2(\omega) = i\mu_0 \omega \sigma_0. \]  

(4.53)
Then we find
\[ k_I \equiv \Im k = \Im i \mu_0 \omega \sigma_0 = \frac{1}{c} \sqrt{\frac{\sigma_0 \omega}{2 \varepsilon_0}}. \]  
(4.54)

The inverse of \( k_I \) is known as skin depth
\[ \delta = \frac{1}{k_I} = c \sqrt{\frac{2 \varepsilon_0}{\sigma_0 \omega}}. \]  
(4.55)

The skin depth determines the distance of penetration of the EM wave into the medium before the beam is dissipated: \( E \propto E_0 e^{-\gamma \delta} \). In practice, \( \delta \) has a very small value (for Cu at \( \omega \sim 10^7 \) s\(^{-1}\), \( \delta \approx 100 \mu m \)), indicating that an optical beam incident on a metallic specimen penetrates only a short distance below the surface.

b) The high-frequency region \( \omega \tau \gg 1 \). This region covers the visible and ultraviolet ranges. Equation (4.49) shows that the dielectric constant is real
\[ \varepsilon = \varepsilon_n - \frac{n e^2}{\omega^2 \tau} = \varepsilon_n - \frac{n e^2}{\omega^2 m} = \varepsilon_n \left(1 - \frac{\omega_p^2}{\omega^2}\right), \]  
(4.56)
where
\[ \omega_p^2 = \frac{n e^2}{\varepsilon_n m}. \]  
(4.57)

The frequency \( \omega_p \) is known as the plasma frequency.

We can see from Eq.(4.56) that the high-frequency region can now be divided into two subregions: In the subregion \( \omega < \omega_p \) and therefore \( \varepsilon < 0 \). In this case the real part of the complex index of refraction is zero:
\[ \Re(n) = \Re(\sqrt{\varepsilon}) = 0. \]  
Since the reflection coefficient (for normal incidence) is given by \( R = \left|\frac{1-n}{1+n}\right| \), we find that in this case to \( R = 1 \). That is, the metal exhibits perfect reflectivity. In the higher subregion \( \omega > \omega_p \) and \( \varepsilon > 0 \), and hence \( k_I = 0 \). In this range, therefore, \( \alpha = 0 \) and \( 0 < R < 1 \), and the metallic medium acts like a non-absorbing transparent dielectric, e.g., glass.

\[ \text{Fig. 4.2 The plasma reflection edge.} \]

Figure 4.2 illustrates the dependence of reflectivity on frequency, exhibiting the dramatic discontinuous drop in \( R \) at \( \omega = \omega_p \), which has come to be known as the plasma reflection edge. The frequency \( \omega_p \) as seen from (4.57) is proportional to the electron density \( n \). In metals, the densities are such that \( \omega_p \) falls into the high visible or ultraviolet range (Table 1).
Table 1: Reflection edges (plasma frequencies) and corresponding wavelengths for some metals

<table>
<thead>
<tr>
<th></th>
<th>Li</th>
<th>Na</th>
<th>K</th>
<th>Rb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_p (10^{16} \text{s}^{-1})$</td>
<td>1.22</td>
<td>0.89</td>
<td>0.593</td>
<td>0.55</td>
</tr>
<tr>
<td>$\lambda_p (\text{Å})$</td>
<td>1550</td>
<td>2100</td>
<td>3150</td>
<td>3400</td>
</tr>
</tbody>
</table>

The nature of this charge density wave, known as a plasma oscillation, can be understood in terms of a very simple model. Imagine displacing the entire electron gas, as a whole, through a distance $x$ with respect to the fixed positive background of the ions (Figure 4.3). The resulting surface charge gives rise to an electric field of magnitude $E = \frac{\sigma}{\varepsilon_B}$, where $\sigma = n_e x$ is the charge per unit area at either end of the slab. This field tends to restore the electron distribution to equilibrium.

![Fig. 4.3 Simple model of a plasma oscillation.](image)

The equation of motion of the electron gas as a whole is

$$m \frac{d^2 x}{dt^2} = -e \frac{n_e x}{\varepsilon_B} = -\frac{ne^2 x}{\varepsilon_B}$$  \hspace{1cm} (4.58)

which lead to oscillation at the plasma frequency $\omega_p^2 = \frac{ne^2}{\varepsilon_B m}$.

**Reflection from a conducting surface**

Now we consider what happens if the EM wave is reflected from a surface of conducting material. We assume that the $xy$ plane forms the boundary between a non-conducting linear medium (1) and a conductor (2), as is shown in Fig.4.4. For simplicity we consider normal incidence.

![Fig.4.4](image)
A plane wave of frequency $\omega$, traveling in the $z$ direction and polarized in the $x$ direction, approaches the interface from the left as is shown in Fig.4.4:

$$E_i(z, t) = E_0 e^{i(k_2 z - \omega t)} \hat{x}$$  
(4.59)

$$B_i(z, t) = \frac{1}{v_i} E_0 e^{i(k_2 z - \omega t)} \hat{y}.$$  
(4.60)

It gives rise to the reflected wave

$$E_r(z, t) = E_0 e^{i(k_2 z - \omega t)} \hat{x}$$  
(4.61)

$$B_r(z, t) = -\frac{1}{v_i} E_0 e^{i(k_2 z - \omega t)} \hat{y},$$  
(4.62)

which travels back to the left in medium (1), and a transmitted wave

$$E_t(z, t) = E_0 e^{i(k_2 z - \omega t)} \hat{x}$$  
(4.63)

$$B_t(z, t) = \frac{k_z}{\omega} E_0 e^{i(k_2 z - \omega t)} \hat{y},$$  
(4.64)

which continues on the right in medium (2). Note that compared to the case of non-conducting medium we have in eq.(4.64) a complex number $k_2/\omega$ instead of a real $1/v_2$. Now we use boundary conditions to find the amplitudes. In the absence of surface changes and currents they have the form:

$$\varepsilon_1 E_i^\parallel = \varepsilon_2 E_t^\parallel$$  
(4.65)

$$B_i^\parallel = B_t^\parallel$$  
(4.66)

$$E_i^\perp = E_t^\perp$$  
(4.67)

$$B_i^\perp = B_t^\perp$$  
(4.68)

At $z = 0$, the combined fields on the left, $E_i + E_r$ and $B_i + B_r$, must join the fields on the right, $E_t$ and $B_t$, in accordance with these boundary conditions. In this case there is no components perpendicular to the surface, so (4.65) and (4.66) are trivial. Eqs. (4.67) and (4.68) require that

$$E_{0i} + E_{0r} = E_{0t}$$  
(4.69)

$$\frac{1}{v_i} (E_{0i} - E_{0r}) = \frac{k_z}{\omega} E_{0t}$$  
(4.70)

Eq. (4.70) can be rewritten as follows

$$E_{0i} - E_{0r} = \beta E_{0i}$$  
(4.71)

where now $\beta$ is complex:

$$\beta = v_i \frac{k_z}{\omega}.$$  
(4.72)

Equations (4.69) and (4.71) are easily solved for the outgoing amplitudes, in terms of the incident amplitude:

$$E_{0r} = \frac{1 - \beta}{1 + \beta} E_{0i}, \quad E_{0t} = \frac{2}{1 + \beta} E_{0i}$$  
(4.73)

These equations are formally identical to those we obtained for the reflection at the boundary between nonconductors, but the resemblance is deceptive since $\beta$ now is a complex number. For a perfect conductor $\sigma = \infty$ and consequently, according to eq.(4.51) $k_2 = \infty$, so $\beta$ is infinite and

$$E_{0r} = -E_{0i}, \quad E_{0t} = 0.$$  
(4.74)
Propagation of wave packets

So far we have considered monochromatic waves. In practice EM waves represent a superposition of monochromatic waves with different frequencies. In this section we consider propagation of such waves in dispersive media. We assume, for simplicity, that $\varepsilon(\omega)$ is real, i.e. we work away from resonances and ignore dissipative effects. We consider a one-dimensional case and let $k(\omega)$ be the positive solution to Eq. (4.30).

A wave packet is obtained by superposing solutions to Eqs. (4.29) and (4.32) with different frequencies $\omega$. Assuming that the waves propagate along the $z$ axis and including both left- and right-moving waves, we have

$$
\psi(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ a_+(\omega)e^{ik(\omega)z} + a_-(\omega)e^{-ik(\omega)z} \right] e^{-i\omega t} d\omega
$$

(4.75)

where $\psi(z,t)$ is the wave packet representing either electric or magnetic component of the EM wave, and field $a_\pm(\omega)$ are complex amplitudes that determine the shape of the wave packet. Alternatively, and more conveniently for our purposes, we can instead superpose solutions with different wave numbers $k$. Defining $\omega(k)$ to be the positive solution to Eq. (4.30), we can write

$$
\psi(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ A_+(k)e^{i\omega(k)t} + A_-(k)e^{-i\omega(k)t} \right] e^{ikz} dk
$$

(4.76)

and it is not too difficult to show that the representation of Eq.(4.75) is equivalent to that of Eq. (4.76). The complex amplitudes $A_\pm(k)$ are determined by the initial conditions we wish to impose on $\psi(z,t)$ and its time derivative. It follows from Eq.(4.76) that

$$
\psi(z,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ A_+(k) + A_-(k) \right] e^{ikz} dk
$$

(4.77)

and Fourier's inversion theorem gives

$$
A_+(k) + A_-(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(z,0)e^{-ikz}dz
$$

(4.78)

On the other hand, differentiating Eq. (4.76) with respect to $t$ produces

$$
\dot{\psi}(z,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega(k)[A_+(k) - A_-(k)] e^{ikz} dk ,
$$

(4.79)

and inversion gives

$$
i\omega(k)[A_+(k) - A_-(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \dot{\psi}(z,0)e^{-ikz}dz .
$$

(4.80)

Solving for $A_+(k)$, we finally obtain

$$
A_+(k) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \psi(z,0) \pm \frac{1}{i\omega(k)}\dot{\psi}(z,0) \right] e^{-ikz}dz
$$

(4.81)

The procedure to follow to construct a wave packet is therefore to choose an initial configuration by specifying $\psi(z,0)$ and $\dot{\psi}(z,0)$, then calculate $A_+(k)$ using Eq. (4.81), and finally, obtain $\psi(z,t)$ by evaluating the integral of Eq.(4.76).
In the following discussion we will choose for simplicity the initial configuration to be *time-symmetric*, in the sense that

\[ \psi(z,0) = 0 \]  

(4.82)

Equation (4.81) then simplifies to

\[ A(k) \equiv A_0(k) = \frac{1}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(z,0) e^{-ikz} dz \]  

(4.83)

and we have an equal superposition of left- and right-moving waves.

**Propagation without dispersion**

Let us first see what the formalism of the preceding subsection gives us when there is no dispersion, that is, when \( n = ck / \omega \) is a constant independent of \( k \). We then have

\[ \omega(k) = \frac{c}{n} k, \]  

(4.84)

where \( v_p = c/n \) is the wave’s phase velocity. In this case, for time symmetric initial configuration, Eq. (4.76) reduces to

\[ \psi(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \left[ e^{ikz} + e^{-ikz} \right] e^{ikz} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \left[ e^{ik(z+v_p t)} + e^{ik(z-v_p t)} \right] dk \]  

(4.85)

But from Eq.(4.83) we have

\[ A(k) = \frac{1}{2 \sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(z',0) e^{-ikz'} dz'. \]  

(4.86)

Substituting this to eq. (4.85) gives

\[ \psi(z,t) = \frac{1}{2} \int_{-\infty}^{\infty} d\zeta \psi(z',0) \int_{-\infty}^{\infty} \left[ e^{ik(z+v_p t-z')} + e^{ik(z-v_p t-z')} \right] dk = \]  

(4.87)

\[ = \frac{1}{2} \int_{-\infty}^{\infty} d\zeta \psi(z',0) \left[ \delta(z+v_p t-z') + \delta(z-v_p t-z') \right] \]

or

\[ \psi(z,t) = \frac{1}{2} \psi(z+v_p t,0) + \frac{1}{2} \psi(z-v_p t,0) \]  

(4.88)

The first term on the right-hand side is (half) the initial wave packet, translated in the \( z \) direction by \( +v_p t \); it represents that part of the wave packet that travels undisturbed in the negative \( z \) direction with a speed \( v_p \). The second term, on the other hand, is the remaining half of the wave packet, which travels toward the positive \( z \) direction with the same speed \( v_p \).

The main features of wave propagation without dispersion are that the wave packet travels with the phase velocity \( v_p = c/n \) and that its shape stays unchanged during propagation.

**Propagation with dispersion**

Let us now return to the general case of propagation with dispersion. Here \( \omega(k) \) is a nonlinear function of \( k \), and for time-symmetric initial configuration we have

\[ \psi(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \left[ e^{i\omega(k)t} + e^{-i\omega(k)t} \right] e^{ikz} dk \]  

(4.89)
To be concrete, suppose that $A(k)$ is sharply peaked around $k = k_0$, with $k_0$ some arbitrary wave number, as is shown in Fig. 4.5. Suppose further that $\alpha(k)$ varies slowly near $k = k_0$, so that it can be approximated by

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk} \bigg|_{k=k_0} (k-k_0) = \left(\omega_0 - \nu_g k_0\right) + \nu_g k,$$

(4.90)

where we denoted $\omega_0 \equiv \omega(k_0)$ and

$$v_g \equiv \frac{d\omega}{dk} \bigg|_{k=k_0}.$$

(4.91)

$v_g$ is called the group velocity. Then the wave packet can be expressed as follows

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \left[ e^{i\left(\omega_0 - \nu_g k_0\right)t} e^{ikz} + e^{i\left(k-k_0\right)t} e^{-i\left(\omega_0 - \nu_g k_0\right)t} e^{-ikz} \right] dk =$$

$$= e^{i\left(\omega_0 - \nu_g k_0\right)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikz} dk + e^{-i\left(k-k_0\right)t} e^{-i\left(\omega_0 - \nu_g k_0\right)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{-ikz} dk$$

(4.92)

or

$$\psi(z,t) = e^{i\left(\omega_0 - \nu_g k_0\right)t} \psi(z + \nu_g t, 0) + e^{-i\left(k-k_0\right)t} e^{-i\left(\omega_0 - \nu_g k_0\right)t} \psi(z - \nu_g t, 0).$$

(4.93)

From Eq.(4.93) we see that in the presence of dispersion, the wave packet travels with the group velocity $v_g$ instead of the phase velocity $v_p$. The phase factors indicate that the shape of the wave suffers a distortion during propagation. These are the main features of wave propagation in a dispersive medium. \(^1\)

If the dispersion relation is expressed as $k(\omega) = \frac{n(\omega)\omega}{c}$, in terms of the index of refraction $n(\omega)$, then

$$\frac{dk}{d\omega} = \frac{1}{c} \left[ n + \frac{dn}{d\omega} \right]$$

and

\(^{1}\) See Sec. 7.9 in Jackson for the illustration of the propagation of Gaussian wave packet in a dispersive medium.
\[ \frac{v_g}{c} = \frac{d \omega}{dk} = \frac{1}{n + \omega \frac{dn}{d \omega}}. \] (4.94)

For example consider a conducting medium in the high-frequency region \((\omega \tau \gg 1)\) above the plasma frequency so that \(\varepsilon\) is real. In this case

\[ n = \frac{\varepsilon}{\varepsilon_0} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \] (4.95)

and

\[ \frac{dn}{d \omega} = \frac{1}{n} \frac{\omega_p^2}{\omega^2}. \] (4.96)

Using eq. (4.94) we find

\[ \frac{v_g}{c} = \frac{1}{n + \omega \frac{dn}{d \omega}} = n. \] (4.97)

Since \(n < 1\), we see that the group velocity \(v_g = cn < c\), whereas the phase velocity \(v_p = c/n > c\).

**Nonlocality in time in the connection between \(D\) and \(E\)**

Another important consequence of the frequency dependence of \(\varepsilon(\omega)\) is a temporally nonlocal connection between the displacement \(D(\mathbf{r}, t)\) and the electric field \(E(\mathbf{r}, t)\). According to eq. (4.20) the monochromatic components of frequency \(\omega\) are related by

\[ D(\mathbf{r}, \omega) = \varepsilon(\omega)E(\mathbf{r}, \omega). \] (4.98)

The dependence on time can be constructed by Fourier superposition. Treating the spatial coordinate as a parameter, the Fourier integrals in time and frequency can be written as follows

\[ D(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int D(\omega, \mathbf{r})e^{-i\omega t} d\omega, \] (4.99)

\[ D(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int D(\mathbf{r}, t)e^{i\omega t} dt, \] (4.100)

with similar expressions for \(E\). The substitution of Eq. (4.99) for \(D(\mathbf{r}, \omega)\) in (4.99) gives

\[ D(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \varepsilon(\omega)E(\mathbf{r}, \omega)e^{-i\omega t} d\omega. \] (4.101)

By inserting the Fourier representation of \(E(\mathbf{r}, \omega)\) into the integral we obtain

\[ D(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \varepsilon(\omega) \left[ \int \frac{1}{\sqrt{2\pi}} \int E(t', \mathbf{r'})e^{i\omega t'} dt' \right] e^{-i\omega t} d\omega = \int \frac{1}{\sqrt{2\pi}} \int \varepsilon(\omega) e^{-i\omega(t-t')} d\omega \int E(t', \mathbf{r'}) dt'. \] (4.102)

Now we introduce a frequency dependent electric susceptibility \((\mathbf{P} = \varepsilon_0 \chi_\varepsilon \mathbf{E})\)

\[ \chi_\varepsilon(\omega) \equiv \frac{\varepsilon(\omega)}{\varepsilon_0} - 1, \] (4.103)

so that the integral (4.102) can be rewritten in the form
\[ \mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t') \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega \right] dt' - \varepsilon_0 \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t') \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_e(\omega)e^{-i\omega(t-t')} d\omega \right] dt' \] (4.104)

If we now take into account
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega = \delta(t-t') \] (4.105)

and define
\[ G(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_e(\omega)e^{-i\omega \tau} d\omega, \] (4.106)

we find
\[ \mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{-\infty}^{\infty} G(\tau) \mathbf{E}(\mathbf{r}, t-\tau) d\tau \] (4.107)

Equation (4.107) gives a nonlocal connection between \( \mathbf{D} \) and \( \mathbf{E} \), so that \( \mathbf{D} \) at time \( t \) depends on the electric field at times other than \( t \). If \( \varepsilon(\omega) \) is independent of \( \omega \), Eq. (4.106) yields \( G(\tau) \propto \delta(\tau) \) and the instantaneous connection is obtained. However, if \( \varepsilon(\omega) \) varies with \( \omega \), \( G(\tau) \) is nonvanishing for some values of \( \tau \) different from zero, and the relation between \( \mathbf{D} \) and \( \mathbf{E} \) becomes nonlocal.