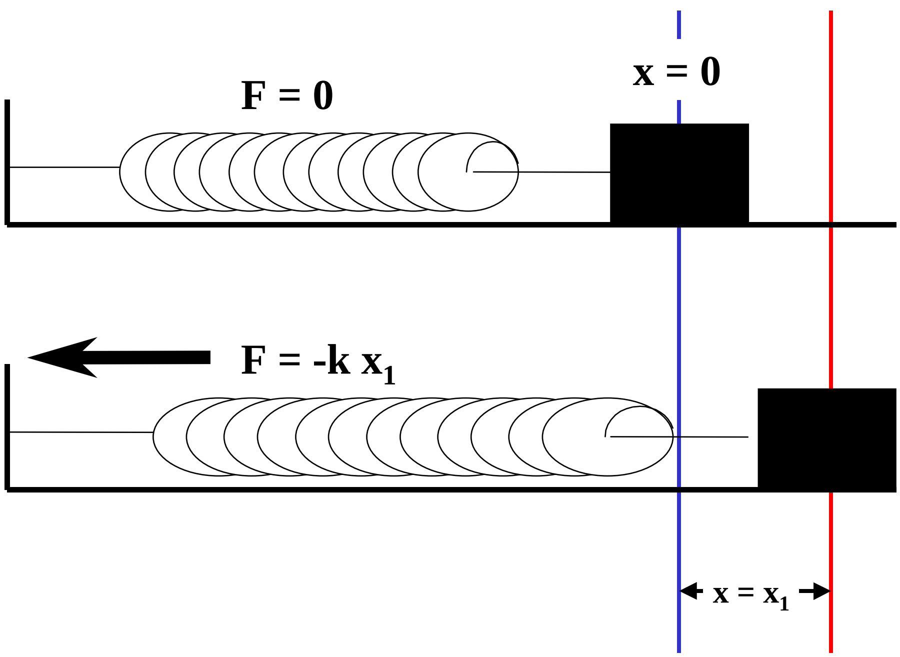
PHYSICS 151 – Notes for Online Lecture #23

Periodicity

Periodic means that something repeats itself. For example, every twenty-four hours, the Earth makes a complete rotation. Heartbeats are an example of periodic behavior. If you look at heartbeats on an electrocardiogram, they make a regular pattern. The pattern that the heart obeys is rather complicated. In this section, we're going to be dealing with a specific type of periodic motion called **simple harmonic motion**

Harmonic means that the motion can be described using sines and cosines.

Simple means that the motion can be described using a single frequency.



A mass on a spring (horizontal or vertical) is a good example of simple harmonic motion (or SHM for short). The motion of the spring is repeated over and over. Let's start with a horizontal spring, resting on a frictionless table.

We pick a reference point on the mass –for example, the center of the mass. The position of the center of the mass when the spring is unstretched is called the '**equilibrium point'** (x = 0). Now I pull the mass an arbitrary distance **x** to the right. The spring exerts a force in the direction opposite the displacement (to the left in this case). The force is given by Hooke's Law:

$$F = -kx$$

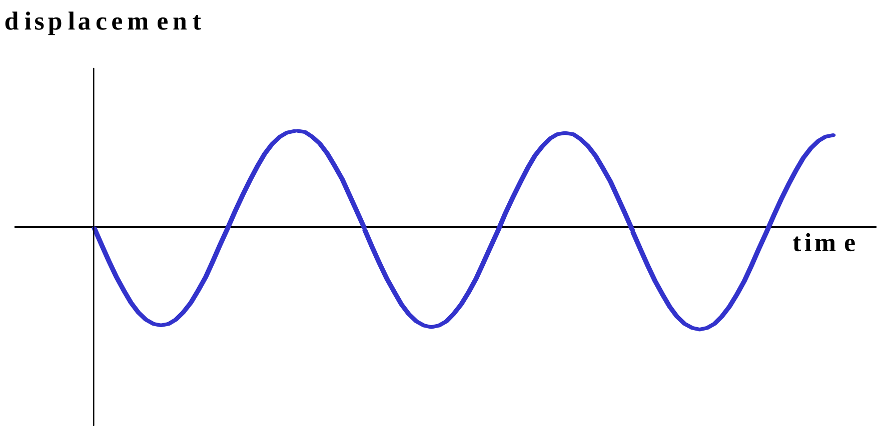
where k is the spring constant and has units of N/m. If I pull the spring to the right, the spring exerts a force to the left. Alternately, I can push the spring in a distance x. Now the spring exerts a force toward the *right*. Remember that Hooke's law only works when the displacements are small. If you make a very large displacement, Hooke's law doesn't apply anymore and none of what I'm about to tell you will apply either.

A special characteristic of simple harmonic motion is that the acceleration is directly proportional to the displacement. We can start with Newton's second law F = ma, and then insert Hooke's law for the force on the spring.

$$a = \frac{F}{m}$$
$$a = \frac{-kx}{m}$$
$$a = -\frac{k}{m}x$$

Any system in which the acceleration is proportional to the displacement will exhibit simple harmonic motion. This can be tested experimentally. Plot F vs. x on a graph and take the slope of the resulting straight line. If you do this and you don't get a straight line, it means that the spring can't be described by Hooke's law.

Simple Harmonic Motion Vocabulary



When I pull the mass on a spring and release it, the mass exhibits a periodic motion – the position of the spring constantly repeats itself. If I were to plot the displacement of the mass as a function of time, it would look something like this:

If you want to find out where the mass is at any point in time, you follow the x-axis out to the time you're interested in and they move up to the curve to see where the mass's position is.

We can define a number of characteristics of simple harmonic motion.

For example, the **amplitude** is the maximum displacement of the mass. The symbol for amplitude is x_o. This is a distance, so the units should be meters.

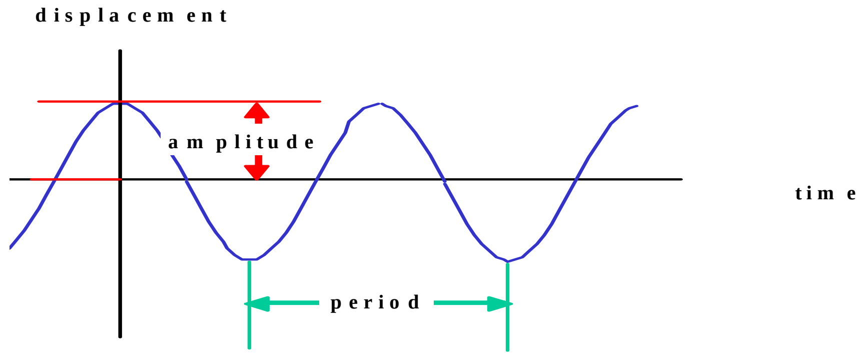
The time it takes for the mass to make one complete cycle – that is, to go from stretched to compressed and back again – is called the **period**, which we represent by 'T'. Remind yourself that the "picture" of the wave is a picture of the mass as a function of time. It's not a snapshot of the wave itself.

The **frequency** is the number of cycles that are completed in one second. The frequency is given by
$$f = \frac{1}{T}$$

If the mass takes 3.0 s to complete a cycle, the frequency is 1/3.0 = 0.33 (1/s). We have a special name for the unit of frequency, which is the Hertz (Hz).

$$Hz = \frac{1}{s}$$

Quantity	Symbol	Definition	Units
Period	T	time for one cycle	s
Frequency	f	number of cycles per second	1/s = Hz
Amplitude	x_o	maximum displacement	m



Describing SHM using sines or cosines

The graph of the wave we have been diagramming can be expressed as a sine or cosine wave. In general, we can write any SHM as a sine wave or a cosine wave. You know from trig that the sine and cosine waves have periods of 2π. The argument of the trig function has to be multiplied by a factor such that the period of your wave is a multiple of 2π. The scale factor turns out to be t/T. When the time is equal to one period, you want your wave to be back where it started. At t = T, the argument is equal to 2π.

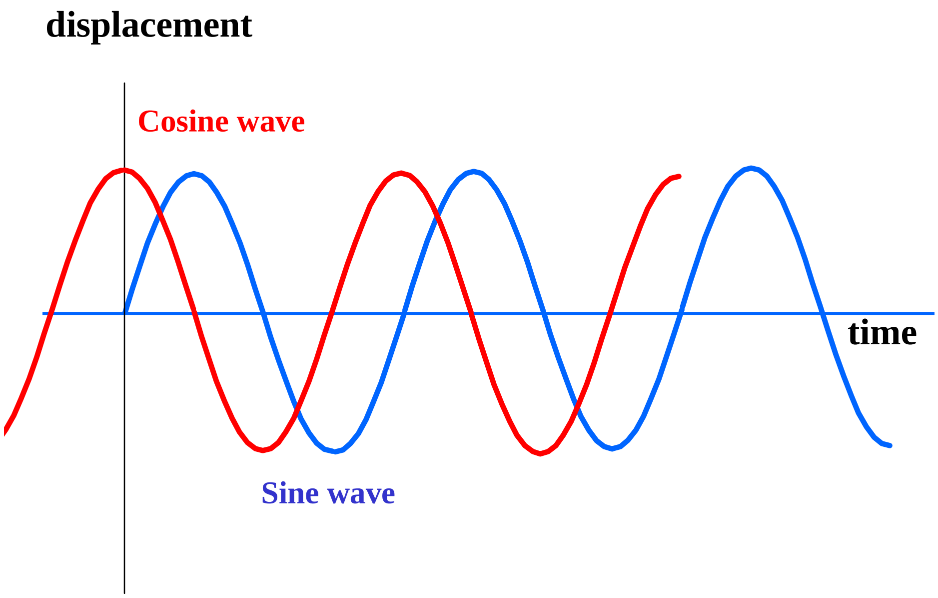
$$x(t) = x_o \cos(2\pi\, ?)$$

$$x(t) = x_o \cos\left(2\pi \frac{t}{T}\right)$$

If you take a cosine wave and shift it by one quarter of a cycle (90 degrees or $\pi/2$ radians), you find that the result is a sine wave.

$$x(t) = x_o \cos\left(\frac{2\pi t}{T} - \frac{\pi}{2}\right) = x_o \sin\left(\frac{2\pi t}{T}\right)$$

How do you know which is which? The answer is that you have to figure out how the wave starts. For example, at t =0, the sine function will always be zero, regardless of the value of omega. The wave below in blue must be a sine wave because it starts at zero.



We can also find the velocity and the acceleration of the mass as a function of time. If

$$x(t) = x_o \cos\left(\frac{2\pi t}{T}\right)$$

then

$$v(t) = -v_o \sin\left(\frac{2\pi t}{T}\right)$$

and

$$a(t) = -a_o \cos\left(\frac{2\pi t}{T}\right)$$

Note that our constraint that x and a must be proportional to each other is satisfied by these expressions.

Ex. 32-1: The motion of an oscillator of mass 0.2 kg is given by:

$$x(t) = (0.50\,m)\cos(2.09t) \quad \text{where x is in m and t is in s}$$

a) Find the amplitude

b) Find the period

c) Find the frequency of oscillation

d) Find the position of the mass at t = 0 s, 0.75 s,1.5 s, 3.0 s and 6.0 s

We first have to put this in the same form -as $x(t) = x_o \cos\left(2\pi \frac{t}{T}\right)$

This give us

$$x(t) = (0.5m)\cos\left(2\pi \frac{t}{3.0}\right)$$

a) $x_o = 0.50$ m

b) The argument in the cosine function is $\frac{2\pi t}{T}$. The period must therefore be 3.0 s.

c) f = 1/T = 1/3.0 s = 0.33 Hz

Time	**2 π t/T**	$\cos\left(2\pi \frac{t}{3.0}\right)$	**X(t)**
0	0	1	0.5 m
0.75 s	π /2	0	0
1.5 s	π	-1	-0.5 m
3.0	2 π	1	0.5 m

Ex. 32-2: A 0.50-kg mass at the end of a horizontal spring has position 0 when t = 0. The amplitude is 0.15 m and the cycle starts by moving to the right first. The mass makes 2.0 complete oscillations each second. What is the equation for the position as a function of time?

Solution:

The function will be either a sine or a cosine. How do we know which to pick? We're told that the position at t = 0 is x = 0. Compare the cosine and sin functions.

function	t = 0	value
$\cos\left(2\pi\frac{t}{T}\right)$	cos(0)	1
$\sin\left(2\pi\frac{t}{T}\right)$	sin(0)	0

So anytime that the mass starts from x =0, you will have a sin function. If the mass starts from its amplitude value, x_0, you need to have a cosine function. Since we're starting from 0, we need to use a sin function.

$$x(t) = x_o \sin\left(\frac{2\pi t}{T}\right)$$

We are told that the system completes two oscillations every second. This is the frequency, f

$$f = 2\ 1/s$$

The period, T, is given by T = 1/f = 0.5 s

The amplitude is given to us as x_o = 0.15 m. Putting these in our equation, we have:

$$x(t) = (0.15\,m)\sin\left(\frac{2\pi t}{0.5}\right) = (0.15\,m)\sin(4\pi t)$$

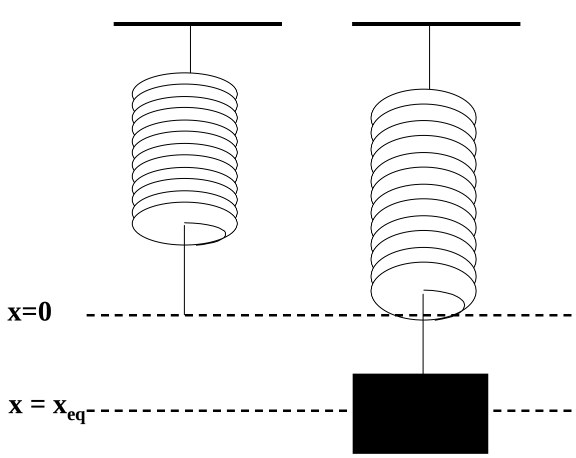
Why were we told that the oscillations started toward the right? So that we would know whether we needed a positive or a negative sign out front. When the mass starts at zero, it can go either positive or negative in displacement. If we take to the right as positive, the equation will not need a negative sign. If the mass were going to the left, we would have a negative sign out front.

The vertical spring

What if the spring you have is hung vertically instead of horizontally? Does what we just discovered still hold?

Ex. 32-3: A spring of spring constant k = 25 N/m has a mass of 0.5 kg hung from it. How far does the spring stretch when the mass is placed on it?

When the mass is on the spring, it pulls the spring down, but then it just hangs there. We can draw a free-body diagram for the mass. The acceleration is zero, and the only forces acting are gravity down and the force of the spring up.



$$\Sigma F = 0$$

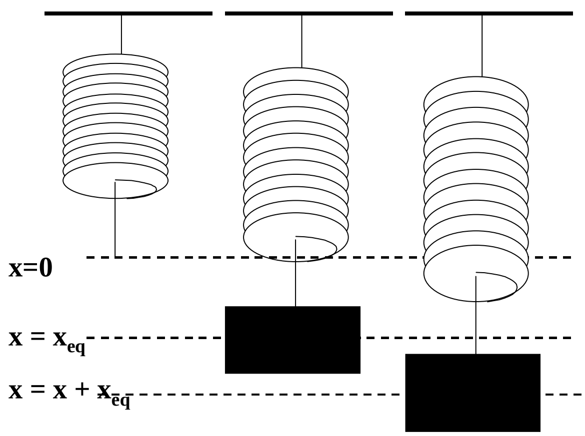
$$mg - kx_{eq} = 0$$

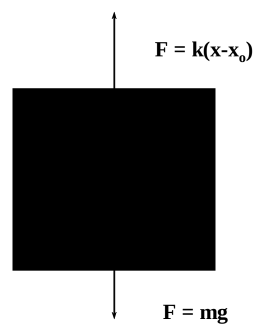
$$x_{eq} = \frac{m}{k} g$$

$$x_{eq} = \frac{(0.5kg)}{25 N/m} 9.8 \tfrac{m}{s^2}$$

$$x_{eq} = 0.196m = 0.20 m$$

This is where the effect of gravity come in- it shifts the equilibrium position of the spring. Once this has been accounted for - by taking the potential energy to be zero when the mass is at $x_{o.}$ Gravity has no effect on the SH motion at all. Let's look at the spring when it's displaced a distance x. Draw the free-body diagram.





The net force is

$$F = k(x + x_{eq}) - mg$$

We found in part a that x_{eq} = mg/k.

$$F = k\left(x + \frac{mg}{k} \right) - mg$$

$$F = k\left(x + \frac{mg}{k} \right) - mg$$

$$F = kx + mg - mg$$

$$F = kx$$

The only force causing the SHM is the spring!

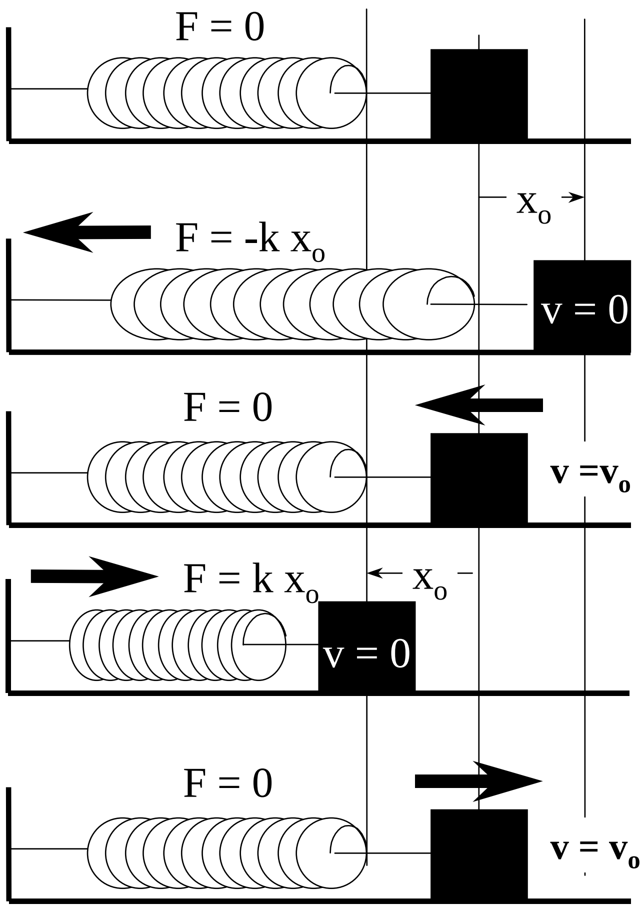
So analyzing SHM in the vertical and the horizontal directions is the same, except that the equilibrium position shift must be accounted for.

Conservation of Energy for SHM

As the spring is stretched or compressed, energy is converted from the motion of the mass and spring to energy stored in the coils and back again. The elastic potential energy due to a spring (and other stretchy things like rubber bands) is:

$$PE_{el} = \frac{1}{2}kx^2$$

where, unlike gravitational potential energy, we take the zero to be the equilibrium position of the spring (i.e. x = 0 corresponds to the point at which there is zero potential energy).



We can write the total mechanical energy for a spring as:

$$E = KE + PE$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At the maximum displacement ($x = x_o$), the mass is momentarily standing still. The total energy is then:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = \frac{1}{2}kx_0^2$$

All potential energy!

When x = 0, the mass has a velocity v_o, which is the maximum velocity that the mass can have. The total energy is then:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = \frac{1}{2}mv_o^2$$

All kinetic energy

Because the total energy is constant at every place along the motion,

$$\frac{1}{2}mv_o{}^2 = \frac{1}{2}kx_o^2$$

$$v_o{}^2 = \frac{k}{m}x_o^2$$

$$v_o = \sqrt{\frac{k}{m}}x_o$$

There is one other relationship that we will need to use (which can be derived by considering SHM is the projection of circular motion).

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Let's review the definitions and relationships we have

x_o = amplitude = maximum displacement - occurs when v = 0 (A is also used for amplitude)

v_o = maximum velocity – occurs when x = 0

T = period (s)

f = frequency (Hz = 1/s)

k = spring constant (N/m)

Relationships:

$T = 2\pi\sqrt{\frac{m}{k}}$	$T = \frac{1}{f}$	$\frac{1}{2}mv_o^2 = \frac{1}{2}kx_o^2$ $v_o^2 = \frac{k}{m}x_o^2$	$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Ex. 23-4: The motion of an oscillator of mass 0.2 kg is given by:

$$x(t) = (0.50\,m)\cos(2.09t)$$

where x is in m and t is in s. Note that this is the same equation as Example 23-1.

e) Find the spring constant

f) Find the total energy

g) Find the maximum velocity

We first have to put this in the same form as $x(t) = x_o \cos\left(2\pi\frac{t}{T}\right)$

This gives us

$$x(t) = \left(0.5m\right)\cos\left(2\pi\frac{t}{3.0}\right)$$

a) spring constant: We notice first that the period is 3.0 s, so

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T^2 = (2\pi)^2 \frac{m}{k}$$

$$k = (2\pi)^2 \frac{m}{T^2}$$

$$k = (2\pi)^2 \frac{(0.2\,kg)}{(3.0\,s)^2}$$

$$k = 0.88\tfrac{kg}{s^2} = 0.88\frac{N}{m}$$

b) Total energy

$$E = \tfrac{1}{2}kx_o^2$$

$$E = \tfrac{1}{2}\left(0.88\tfrac{N}{m}\right)(0.50\text{m})^2$$

$$E = 1.1\,x\,10^{-1}\,J$$

c) Maximum velocity: The maximum velocity occurs when x = 0, so the energy is entirely kinetic

$$E = \tfrac{1}{2}mv_o^2$$

$$\sqrt{\frac{2E}{m}} = v_o$$

$$\sqrt{\frac{2(1.1\,x\,10^{-1}\,J)}{0.2\,kg}} = v_o$$

$$1.05\tfrac{m}{s} = v_o$$

$$1.1\tfrac{m}{s} = v_o$$

You Try It!

A 0.50-kg mass at the end of a horizontal spring is pulled back to a distance of 0.15 m. At t = 0, the mass is released and makes 3.0 complete oscillations each second. Find:

a) the velocity when the mass passes the equilibrium point

b) the velocity when the mass is 0.10 m from equilibrium

c) the total mechanical energy of the system

Known: A = 0.15 m m = 0.50 kg f = 3.0 Hz

a) The quantity we are looking for is v_o. In examining the equations for velocity and position, we found that

$$v_o = \sqrt{\frac{k}{m}}x_o$$

Unfortunately, we don't know k, but we can find k from

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$\frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$
$$\frac{1}{2\pi f} = \sqrt{\frac{m}{k}}$$
$$\left(\frac{1}{2\pi f}\right)^2 = \frac{m}{k}$$
$$k = (2\pi f)^2 m$$

Now put this in our expression for v_o

$$v_o = \sqrt{\frac{k}{m}}x_o$$
$$v_o = \sqrt{\frac{(2\pi f^2)m}{m}}x_o$$
$$v_o = 2\pi f^2 x_o$$

$$v_o = 2\pi f x_o$$
$$v_o = 2\pi(3.0\,\text{Hz})(0.15\,\text{m})$$
$$v_o = 2.8\,\tfrac{\text{m}}{\text{s}}$$

Using conservation of energy:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv_o^2$$
$$v^2 + \frac{k}{m}x^2 = v_o^2$$
$$v^2 = v_o^2 - \frac{k}{m}x^2$$
$$v^2 = v_o^2 - \frac{(2\pi f)^2 m}{m}x^2$$
$$v^2 = v_o^2 - (2\pi f)^2 x^2$$
$$v = \sqrt{v_o^2 - (2\pi f)^2 x^2}$$
$$v = \sqrt{\left(2.8\,\tfrac{\text{m}}{\text{s}}\right)^2 - \left(2\pi(3.0\,\tfrac{1}{\text{s}})\right)^2(0.10\,\text{m})^2}$$
$$v = 2.1\,\tfrac{\text{m}}{\text{s}}$$

Stop to see if this makes sense. The velocity must be less than v_o, which it is.

c) Total energy

$$E = \tfrac{1}{2} m v_o^2$$

$$E = \tfrac{1}{2}\left(0.5\text{kg}\right)\left(2.8\,\tfrac{\text{m}}{\text{s}}\right)^2$$

$$E = 2.0\text{J}$$