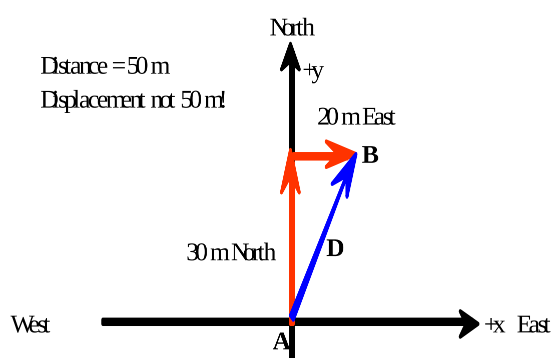
PHYSICS 151 – Notes for Online Lecture #6

Vectors - A vector is basically an arrow. The length of the arrow represents the magnitude (value) and the arrow points in the direction. Many different quantities can be represented by vectors: displacement, velocity, and acceleration are some that we have talked about, and force, momentum, etc. are some that we will get to in the next few weeks. Regardless of what quantity the vector represents, we treat all of the vectors mathematically the same. Vectors are written in boldface in your text and in the notes. On the board, I will write vectors with an arrow over them to remind you that these quantities can't be added and subtracted like ordinary scalar numbers.

There are two ways to describe a vector. For example, let's say that we determine that the magnitude of a velocity vector is 15 m/s. If I wanted to describe this velocity to you, I might say that **V** is a vector of magnitude 15 m/s at an angle +35° from the + x axis. (Remember that angles going counterclockwise from the specified axis are positive and those going clockwise from the specified axis are negative.) This description gives you the **magnitude** (15 m/s) and the **direction** (+35°) of the vector. With those two pieces of information, you can draw the vector I described.

The second way of specifying a vector is to recognize that part of this velocity vector acts horizontally – in the x-direction – and part of the vector acts vertically – in the y-direction. All two-dimensional vectors can be expressed as the sum of a component in the x direction and a component in the y direction.



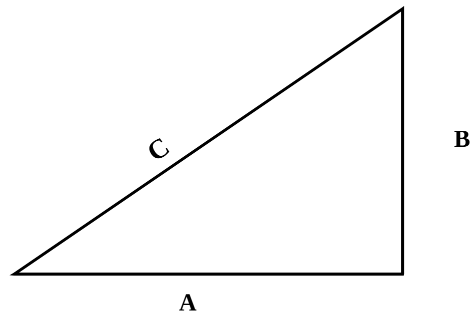
Let's say that you and I are both traveling from point A to point B. We'll put the origin of our coordinate system at point A. The destination, point B, is a point 20 m East and 30 m North of the origin. There are two ways for us to get there. You take a helicopter and fly directly there, taking the path I'll call "**D**" in the figure. I have to drive on the city streets, which run north-south and east-west, so I get there by going 30 m North and then going 20 m East.

Both of us arrive at the same point; however, the paths we took were different. We'll call the first part of my trip, which was entirely in the y-direction, D_y and the second part of my trip, which was in the x-direction, D_x. D_x and D_y are the **components** of the vector **D**. Note that D_x and D_y are themselves vectors – they are special because they point in only the x and y directions respectively.

The magnitude of **D** can be related to the magnitudes of the components that make up the vector **D**. The three vectors make up a right triangle, so we can use the Pythagorean theorem. For a right triangle with sides of length A, B and C, as shown:

$A^2 + B^2 = C^2$, or $C = \sqrt{A^2 + B^2}$

where A, B, and C are the lengths of the sides.



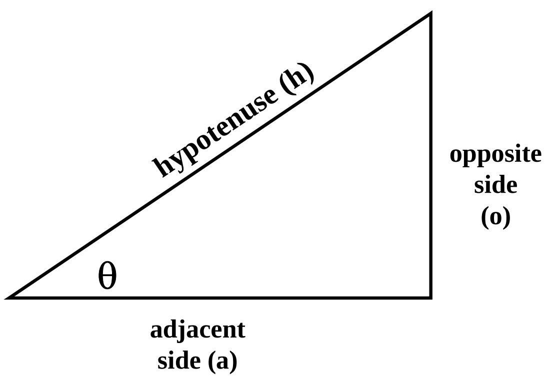
We write the symbol for the magnitude of a vector using the name of the vector, but without a vector sign or any bold. The magnitude of **D** is D.
For our example, the magnitude of the vector **D** is given by:

$$D = \sqrt{D_x{}^2 + D_y{}^2}$$

$$D = \sqrt{(20\,\text{m})^2 + (30\,\text{m})^2}$$

$$D = 36\,\text{m}$$

We can also determine the angle that the vector D makes with some axis. For that, we need to review some trig.



If we pick an angle θ, we call the side nearest to the angle the *adjacent* side and the side across from the angle the *opposite* side. From trig:

$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{o}{h}$$

$$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{a}{h}$$

$$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{o}{a}$$

A mnemonic for the trig functions that some people find useful is:
Soh-cah-toa (sock-a-toe-a), which stands for sine = opposite/hypotenuse, cosine = adjacent/hypotenuse and tangent = opposite/adjacent.
For our example, we have the following triangle. I'm going to choose the angle that the vector **D** makes with the y-axis as θ. Then, from our definitions,

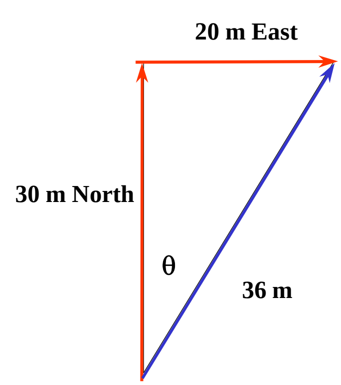
$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{o}{h}$$

$$\sin\theta = \frac{20\,\text{m}}{36\,\text{m}}$$

$$\sin\theta = 0.5555556$$

$$\theta = \sin^{-1}(0.5555556)$$

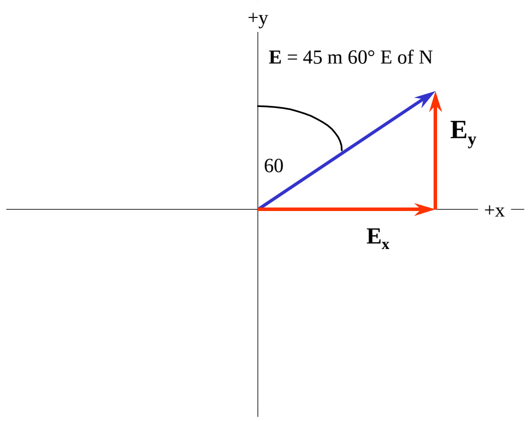
$$\theta = 33.7 = 34°$$



We see from this that the vector **D** can therefore be described in two different, but equivalent ways:
D is 30 m North and 20 m East (or 30 m in the y direction and 20 m in the x direction)
D as magnitude 36 m and direction -34° from the +y axis.
In the latter definition, you might also describe the angle as being 34 degrees East of north

Components

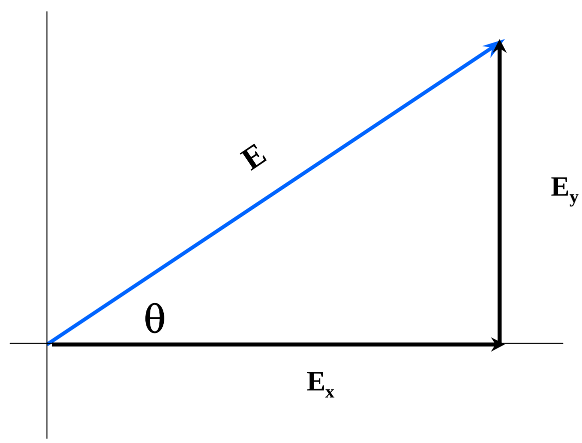
In the last section, we took the components and put them together to make the resultant vector. We can also decompose any general vector into its components. Let's look at a vector: **E** = 45 m at 60 degrees east of north.



The vector **E** can be drawn as the sum of two vectors placed head to tail - one lying along the x-axis and one along the y-axis. These two vectors are the components of **E**. We now have to reverse the procedure we went through before for determining how two perpendicular vectors added up to make the resultant. This is called **decomposing** the vector. We call the components of the vector $\mathbf{E_x}$ and $\mathbf{E_y}$.. Specifying the components of a vector uniquely identifies that vector, just as the magnitude and direction completely describe the vector.

If we re-draw the vector and components as shown below, selecting the angle θ as shown, we see that the components can be determined from:

$$E_x = E\cos(\theta) \Rightarrow \text{adj} = \text{hyp}(\cos\theta)$$
$$E_y = E\sin(\theta) \Rightarrow \text{opp} = \text{hyp}(\sin\theta)$$



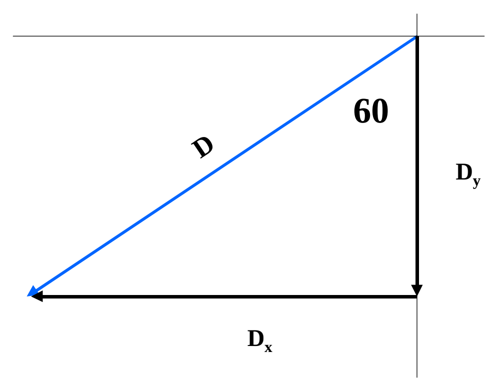
BUT BE VERY CAREFUL! Note that the angle given is 60° east of North. The angle that we've used in our drawing is the complement of 60°. The angle you have to use in the above equations is 90°-60° = 30°

For this case: $E_x = 45\text{ m}\cos(30) = 39\text{ m}$
$E_y = 45\text{ m}\sin(30) = 23\text{ m}$

EXAMPLE 6-1: Find the components of vector **D,** which is 10 m at 60° W of S. Assume that the length is known to two significant figures.

From drawing the vector D,

$$D_x = -10\text{ m}\sin(60) = -8.7\text{ m}$$
$$D_y = -10\text{ m}\cos(60) = -5.0\text{ m}$$



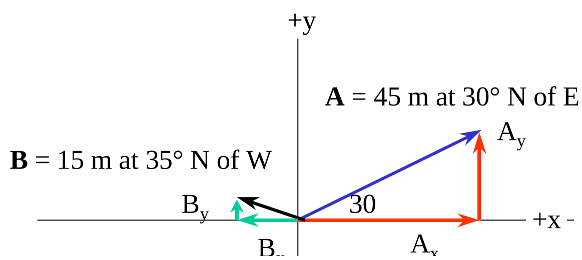
How did I know to put a minus sign? If I'm careful to add the vectors head to tail, starting with the tail of the first vector at the origin, I see that D_x points in the negative x direction and D_y points in the negative y direction.

Anytime you decompose a vector, you have to look at the original vector and make sure that you've got the correct signs on the components.

John walks 0.85 mi south and then 1.35 mi west. What is his displacement from the origin?

Vector Algebra (Addition and Subtraction of Vectors)

Let's say that we want to add a vector of magnitude 15.0 m at an angle of 35° degrees North of West to a vector of 45.0 m at 30° North of East. It is critical to draw the vectors you're adding. The graphical check is helpful in case you mess up the math. Since you can't just add vectors directly, you have to break each vector into its component parts, add the components, and then re-assemble the vector.

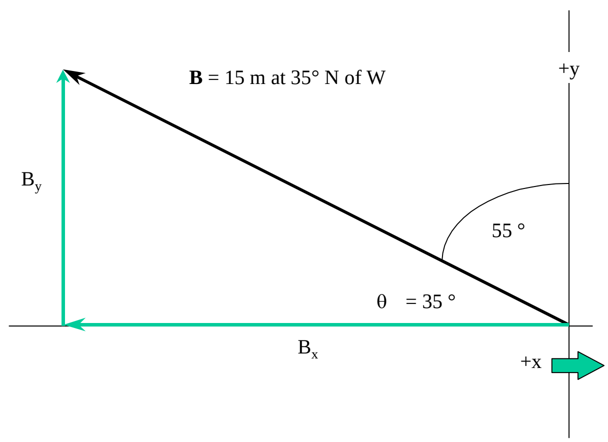


Start with vector **B**, which makes an angle of 35 degrees with the negative x axis.

$\mathbf{B_x} = -B \cos \theta$

The minus sign is important because the vector points toward the negative x axis!

$\mathbf{B_y} = B \sin \theta$



$\mathbf{B_x} = -B \cos \theta = -15 \text{ m} \cos (35) = -12.3 \text{ m}$

$\mathbf{B_y} = B \sin \theta = 15 \text{ m} \sin (35) = 8.57 \text{ m}$

Vector A can also be decomposed.

$\mathbf{A_x} = A \cos(30) = 45 \cos (30) = 39.0 \text{ m}$

$\mathbf{A_y} = A \sin(30) = 45 \sin(30) = 22.5 \text{ m}$

Now, the scalar parts of the components in each direction can be added because they're each in the same direction.

	x-component	y-component
A	39.0 m	22.5 m
B	-12.3 m	8.57 m
A+B = C	26.7 m	31.1 m

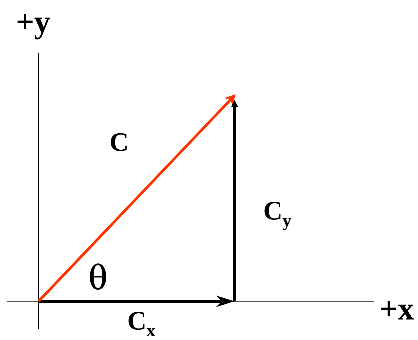
We can re-assemble vector C in magnitude/angle notation:

$$C = \sqrt{C_x^{\ 2} + C_y^{\ 2}} = \sqrt{(26.7\,\text{m})^2 + (31.1\,\text{m})^2}$$

$$C = 41.0\,\text{m}$$

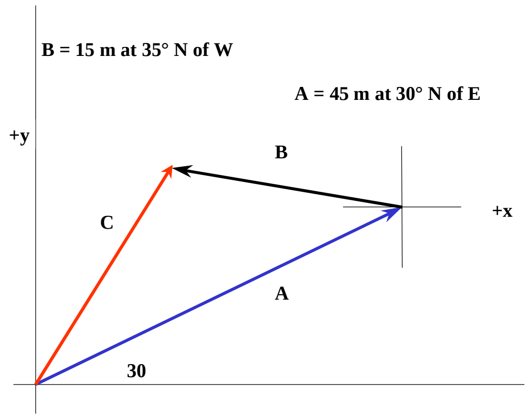
The direction can be found using the trig functions. We know all three of the sides, because we just determined the hypotenuse:

$$\tan\theta = \frac{C_y}{C_x} = \frac{31.1\text{m}}{26.7\text{m}} = 49.3° = 49°$$



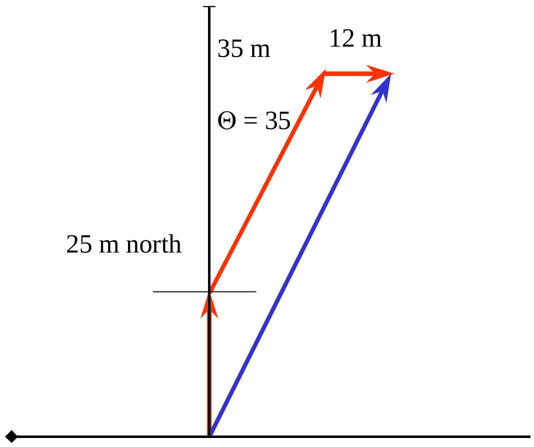
The problem now is that you have to know exactly where this angle is. One way is to draw the x and y components and add them graphically. When you add vectors graphically, you place the vectors together head to tail. So for our components, C_x and C_y, the drawing looks like the drawing at left. From our definition of θ (opposite over adjacent, we see that the angle θ is the angle between the vector and the x-axis.

The vectors **A** and **B** can also be added graphically. To add two vectors graphically, start by drawing the first vector with its tail at the origin. At the tip (or head) of this first vector, draw another coordinate axis and draw the second vector, with the tail of the second vector sitting at the origin of the new coordinate axis. The resultant vector is found by drawing a vector from the tail of the first vector (**A**) to the head of the second vector (**B**), as shown. Graphically adding the vectors is important to ensure that you haven't made a mistake in the mathematics.



EXAMPLE 7-2:
Add the following three vectors:
A = 25.0 m north
B = 35.0 m at 35 degrees east of north
C = 12.0 m east
R = **A** + **B** + **C**

1) draw each vector tip to tail
2) Break each vector into components
   $A_x = 0$ m $A_y = 25.0$ m
   $B_x = 35.0$ m sin (35) = 20 m $B_y = 35.0$ m cos (35) = 28.7 m
   $C_x = 12.0$ m $C_y = 0$ m



3) Add up the x-components and y-components individually

	x-component	**y-component**
A	0 m	25.0 m
B	20.0 m	28.7 m
C	12.0 m	0 m
R	32.0 m	53.7 m

4) Re-compose the vector:

Magnitude: $R^2 = R_x^2 + R_y^2 = (32.0\text{m})^2 + (53.7\text{ m})^2 = 391\text{ m}^2$ so R = 62.5 m

Direction: From trig:

$\tan\theta = R_y/R_x = 59.2°$

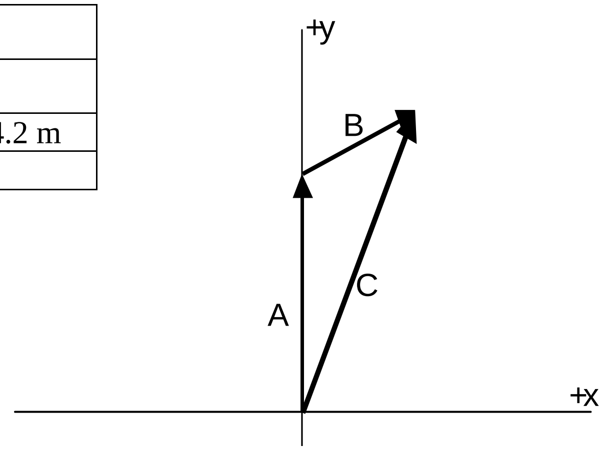
So the resultant vector is length 62.5 m at an angle of 59.2 degrees from the horizontal. Check the graphical addition picture to ensure that the result agrees with the picture.

Subtraction of Vectors

EX 7-3: If A = 20 m due North and B = 10 m at 25 degrees N of E, find A+B and A – B.

A + B

	x-component	**y-component**
A	0 m	20.0 m
B	10.0 m cos (25) = 9.1 m	10.0 m sin (25) = 4.2 m
A+B	9.1 m	24.2 m



Magnitude = $\sqrt{(9.1m)^2 + (24.2\,m)^2} = 26m$

Direction $\theta = \tan^{-1}\left(\frac{24.2\,\mathrm{m}}{9.1\,\mathrm{m}}\right) = 69°$ N of E

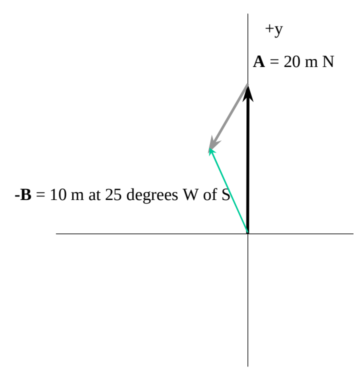
Subtraction is addition of a negative vector:

A – **B** = **A** + (-**B**).

We already know how to add, now we just need to know what **-B** means. **-B** means that the vector is multiplied by -1. The minus sign changes the direction by exactly 180 degrees from what it was. Note that the magnitude doesn't change because all we're doing is multiplying by a factor of one. We can now do the subtraction by adding **-B** to **A**.

Now do subtraction:

	x-component	y-component
A	0 m	20.0 m
-B	-10.0 m cos (25) = - 9.1 m	-10.0 m sin (25) = -4.2 m
A-B	-9.1 m	15.8 m



Magnitude = $\sqrt{(-9.1\,\mathrm{m})^2 + (15.8\,\mathrm{m})^2} = 18.2\,\mathrm{m}$

Direction $\theta = \tan^{-1}\left(\frac{15.8\,\mathrm{m}}{-9.1\,\mathrm{m}}\right) = -60°$ N of W

What does this mean? Make sure you draw the x and y components to identify the correct triangle. The minus sign doesn't provide you with additional information if you draw the picture. Here, the angle is 68.9 degrees N of W. (or 21.1 degrees W of N)

You Try It!

Vector **A** is equal to 35.0 m due west and Vector **B** is equal to 65.0 m 10° W of N. Find **A**+**B** and **A**-**B**.